Optimal tie-breaking rules*

Sumit Goel[†] Amit Goyal[‡]

April 26, 2023
(Link to latest version)

Abstract

We consider two-player contests with the possibility of ties and study the effect of different tie-breaking rules on effort. For ratio-form and difference-form contests that admit pure-strategy Nash equilibrium, we find that the effort of both players is monotone decreasing in the probability that ties are broken in favor of the stronger player. Thus, the effort-maximizing tie-breaking rule commits to breaking ties in favor of the weaker agent. With symmetric agents, we find that the equilibrium is generally symmetric and independent of the tie-breaking rule. We also study the design of random tie-breaking rules that are unbiased ex-ante and identify sufficient conditions under which breaking ties before the contest actually leads to greater expected effort than the more commonly observed practice of breaking ties after the contest.

1 Introduction

Contests are situations in which agents exert costly effort to win one or more prizes. Examples of such competitive situations include sporting contests, promotional tournaments, political contests, R&D races, etc. In many of these situations, it is often the case that there is no outright winner and the contest ends in a draw or a tie. Moreover, a draw may not be an acceptable outcome for the designer. For instance, in sports competitions such as cricket, chess, and soccer, a significant fraction of the games end in a draw. But if these games happen to be knockout games of a world event, the designer must determine a single winner. Many different tie-breaking rules have been used to determine a winner in such situations. The result might be decided by chance[1] or it might be pre-determined based on

*We are grateful to Arunava Sen, Federico Echenqiue, and Wade Hann-Caruthers for helpful comments and suggestions.

[†]California Institute of Technology; sgoel@caltech.edu; 0000-0003-3266-9035

[‡]Indian Statistical Institute Delhi; amit.kr.goyal@gmail.com

[1]The outcome of many elections that ended with ties have been determined via a coin toss (e.x. Kentucky city mayor race 2022). Even in sporting competitions where ties are broken by another short duration contest, like the super over in cricket or the penalty shootout in soccer, it can be argued that the outcome is almost a random draw as there is relatively little scope for skill or effort to have an impact.

some personal attributes like age, sex, height, weight, or status (incumbent or challenger)[2]. In this paper, we consider contests between two agents and focus on understanding how different tie-breaking rules compare in terms of the effort they induce.

We study the effect of tie-breaking rules for three contest environments that differ in how the effort exerted by the players determine the distribution over contest outcomes (player 1 wins, player 2 wins, or tie). We study ratio-form contests (Tullock [51], Baik [4], Ewerhart [20], Wang [53], Nti [46, 47]), difference-form contests (Hirshleifer [30], Baik [3], Skaperdas [49], Beviá and Corchón [6], Che and Gale [9], Ewerhart [21]), and lastly, we discuss examples of concave contests (Blavatskyy [7], Fu and Wu [25]). While the literature has typically assumed only two possible outcomes[3], we will consider generalizations of these contests that allow for the possibility of ties (Blavatskyy [7], Jia [33], Vesperoni and Yildizparlak [52]). A feature of some of these generalizations is that the probability of tie increases as the efforts come closer, and in particular, it is maximized when both agents exert equal efforts. Motivated by this, with ratio-form and difference-form contests, we assume that the probability of a tie increases as the contest becomes more equal (ratio of efforts goes to 1 or difference goes to 0). In addition, we make assumptions that ensure existence and uniqueness of pure-strategy Nash equilibrium.

We make two primary contributions. First, we find that when the two agents differ in their value from winning, their effort is decreasing in the probability that the ties are broken in favor of the stronger player. Thus, an effort-maximizing contest designer would prefer to commit to breaking ties in favor of the weaker player. In this way, our results lend support to the idea of leveling the playing field and increasing the competitive balance of a contest to increase effort. Second, we find that with symmetric agents, the equilibrium is generally symmetric and independent of the tie-breaking rule. We also discuss the design of random unbiased tie-breaking rules and identify conditions under which breaking ties before the contest would lead to greater effort than the standard practice of breaking them after the contest has ended in a tie. For all our results, we identify parametric classes of contests that satisfy our assumptions and discuss the application of our results to them.

There is a growing literature studying the effect of introducing ties on equilibrium effort in contests. For concave contests with ties, introduced by Loury [38] and axiomatized by Blavatskyy [7], the possibility of a tie has been shown to reduce total equilibrium effort (Nti [45], Deng et al. [15], Li et al. [37]) though it may increase winner's expected effort (Deng et al. [15]) or even total effort in a winner-pay setting (Minchuk [43]). In all-pay auctions, finite strategy spaces and bid gaps have been used to study the effect of introucing ties

[2]In weightlifting contests, ties were resolved, until 2016, in favor of the lighter athlete. In boxing, if a championship bout ends in a draw, the champion usually retains the title. In the 1999 Cricket World Cup, the tied semi-final match between Australia and South Africa went to Australia because they had defeated South Africa earlier in the tournament.

[3]Surveys of this literature can be found in Garfinkel and Skaperdas [26], Jia et al. [34], Corchón and Serena [14], Chowdhury et al. [11], Mealem and Nitzan [42]

(Eden et al. [18], Cohen and Sela [12], Gelder et al. [27]). Other related work has illustrated the merits of introducing draws in different contests (Nalebuff and Stiglitz [44], Imhof and Kräkel [31, 32], Chang et al. [8]). In comparison to this literature, a tie is a natural outcome in our contests and the prize is awarded irrespective of the contest outcome.

The paper contributes to the literature studying the effect of draw prizes on effort. Most of the work in this domain assumes symmetric agents and studies the effect of a common draw prize on effort. For a generalization of ratio-form contests to allow for ties, Vesperoni and Yildizparlak [52] find that there is a unique symmetric equilibrium that does not depend on the common draw prize[4]. In ratio-form contests, Jia [33] shows that there is always a unique equilibrium that is symmetric even though the contest might be biased[5]. They illustrate how introducing ties with no prizes can lead to equilibrium that is no longer symmetric. In comparison to this literature, our model allows for asymmetric agents and can be interpreted as studying the effects of awarding complementary draw prizes. We find that for both Vesperoni and Yildizparlak [52] and Jia [33] contests, the equilibrium effort is decreasing in the probability that a tied contest is awarded to the stronger player. In a similar spirit, Szech [50] finds that an asymmetric tie-breaking rule that favors the weaker player increases expected bids in an all-pay auction environment.

The paper also contributes to the literature studying the effect of leveling the playing field on effort in contests with heterogeneous agents. There are many different mechanisms that have been studied including multiplicative biases and additive headstarts[6]. In short, it has been found that if the designer cares about increasing effort and can optimally choose multiplicative biases, it cannot gain from being able to further choose additive biases or introduce draws (Fu and Wu [25], Franke et al. [22], Li et al. [37]). We believe ours is the first to consider this idea of leveling the playing field in the context of tie-breaking rules. We note here that for some contests (like the class of concave contests from Fu and Wu [25]), the choice of a tie-breaking rule is essentially equivalent to defining different head-starts for the two players. However, the tie-breaking rule imposes constraints on the set of feasible head-starts (such as the sum of head-starts must be a constant) and so the problem of finding the optimal tie-breaking rule in these contests is still different from the problem of finding optimal headstarts as considered in these papers.

The paper proceeds as follows. In section 2, we present our model of a two-player contest with ties. In section 3 and 4, we study ratio-form and difference-form contests respectively. In section 5, we discuss some other contest examples and study the design of random unbiased tie-breaking rules. Section 6 concludes. Most of the proofs are relegated to the appendix.

[4]They also find conditions under which the equilibrium effort is greater with the possibility of draw as compared to without it, even though the draw prize may be 0.

[5]This is referred to as the homogeneity paradox as one would probably expect that players put in different levels of effort if the contest favors one player over another.

[6]Surveys of this literature can be found in Mealem and Nitzan [42], Chowdhury et al. [11].

2 Model

There are two risk-neutral players competing in a contest. The contest has three possible outcomes: player 1 wins, player 2 wins, or it may be a tie. The distribution over the three outcomes is determined by the efforts $x_1 \geq 0, x_2 \geq 0$ exerted by the two players. We let $p_i(x_1, x_2)$ denote the probability that player i wins and $p_0(x_1, x_2)$ denote the probability that the contest ends in a tie. Thus, for any $x_1, x_2 \geq 0$, $p_1 + p_2 + p_0 = 1$. In case of a tie, the designer awards the contest to player 1 with probability $q \in [0, 1]$ and awards it to player 2 with remaining probability $1 - q$. We refer to q as the *tie-breaking rule*. Note that q is independent of the effort exerted by the agents. Given the tie-breaking rule q, player 1 eventually wins the contest with probability $P_1 = p_1 + qp_0$ while player 2 eventually wins with probability $P_2 = p_2 + (1-q)p_0$. The value of player $i \in \{1, 2\}$ from winning the contest is V_i, where we assume $V_1 \geq V_2 > 0$. The cost of exerting effort x_i for player i is given by $c(x_i)$ where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a cost function. We will assume either linear $c(x) = x$ or quadratic costs $c(x) = x^2$ in our analysis.

Given a contest $C = (V_1, V_2, p_1(), p_2(), q, c())$, player 1's payoff under profile (x_1, x_2) is

$$\Pi_1(x_1, x_2) = V_1(p_1(x_1, x_2) + qp_0(x_1, x_2)) - c(x_1),$$

and that of player 2 is

$$\Pi_2(x_1, x_2) = V_2(p_2(x_1, x_2) + (1 - q)p_0(x_1, x_2)) - c(x_2).$$

An effort profile (x_1^*, x_2^*) is a pure-strategy Nash equilibrium if it satisfies for both $i \in \{1, 2\}$:

$$\Pi_i(x_i^*, x_{-i}^*) \geq \Pi_i(x_i, x_{-i}^*) \text{ for all } x_i \in \mathbb{R}_+.$$

We will impose conditions on C that guarantee the existence and uniqueness of a pure-strategy Nash equilibrium, characterized by the first-order conditions. We note here the two first-order conditions. The first-order condition for player 1 is

$$\frac{\partial \Pi_1}{\partial x_1} = 0 \implies V_1 \left(\frac{\partial p_1}{\partial x_1} + q \frac{\partial p_0}{\partial x_1} \right) = c'(x_1), \tag{1}$$

and that for player 2 is

$$\frac{\partial \Pi_2}{\partial x_2} = 0 \implies V_2 \left(\frac{\partial p_2}{\partial x_2} + (1 - q) \frac{\partial p_0}{\partial x_2} \right) = c'(x_2). \tag{2}$$

Given $V_1, V_2, p_1(), p_2(), c()$, we consider the designer's problem of choosing a tie-breaking rule $q \in [0, 1]$ so as to maximize the total effort in the pure-strategy Nash equilibrium[7]. Formally, the designer's problem is

$$\max_{q \in [0,1]} x_1(q) + x_2(q)$$

where $x_i(q)$ refers to the equilibrium effort level of player i under the contest given by $C = (V_1, V_2, p_1(), p_2(), q, c())$. Going forward, we will study and solve the designer's problem for different classes of contest games.

[7]We note here that for some of our results, we only need the objective to be an increasing function of the efforts.

3 Ratio-form contest success functions

In this section, we consider instances of our model where the distribution over the three outcomes depends only on the ratio of the efforts $\theta = \dfrac{x_1}{x_2}$ of the two players. Formally, we assume that there exist twice-differentiable functions $p : \mathbb{R}_+ \to [0, 1]$ and $p_0 : \mathbb{R}_+ \to [0, 1]$ such that

- $p_1(x_1, x_2) = p(\theta)$ and $p_2(x_1, x_2) = p(\frac{1}{\theta})$,
- $p_0(x_1, x_2) = p_0(\theta)$ and $p_0(\theta) = p_0(\frac{1}{\theta})$ for all $\theta \in \mathbb{R}_+$,
- $c(x) = x$.

We'll make the following assumption on these functions.

Assumption 1. Suppose $z_q(\theta) = p(\theta) + qp_0(\theta)$.

- For any $q \in [0, 1]$, $z_q'(\theta) > 0, z_q''(\theta) < 0$, and $2z_q'(\theta) + \theta z_q''(\theta) > 0$, for all θ in R_+.
- For any $q \in [0, 1]$, $z_q(0) = 0$ and $\lim_{\theta\to\infty} z_q(\theta) = 1$

Note that $z_q(\theta)$ denotes the probability that player 1 eventually wins the prize and $1 - z_q(\theta)$ is the probability that player 2 wins eventually wins the prize. The first part of the assumption then says that a player's probability of winning is increasing in its own effort, decreasing in the effort of the other player, and in addition, it is increasing at a decreasing rate in one's own effort. The second part of the assumptions says that if a player exerts 0 effort while the other player exerts a positive effort level, its probability of winning is 0 irrespective of the tie-breaking rule q.

The next assumption we make is on the probability of ties $p_0(\theta)$ and is motivated by the idea that the probability of a tie increases as the contest becomes more closely contested.

Assumption 2. The probability of tie $p_0(\theta)$ is increasing for $\theta \in (0, 1]$ and decreasing for $\theta \in [1, \infty)$.

We first provide a characterization of the pure-strategy Nash equilibria of the ratio-form contest game with ties.

Theorem 1. *Consider a ratio-form contest satisfying assumption 1. The contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1\beta z_q'(\beta) \text{ and } x_2^* = V_2\beta z_q'(\beta),$$

where $\beta = \frac{V_1}{V_2} \geq 1$.

To prove the result, we show that there is a unique solution to the first-order conditions 1 and 2 for the contest game defined by x_1^* and x_2^*. The concavity of the payoffs implied by assumption 1 means that the second-order conditions for maximization are satisfied[8]. We note here that the result also follows from the result of Baik [4] who characterized the pure-strategy Nash equilibrium of two player ratio-form contests satisfying assumption 1.

The next result shows that the effort-maximizing tie-breaking rule breaks ties in favor of the weaker agent.

Theorem 2. *Consider a ratio-form contest satisfying assumptions 1 and 2.*

- *If* $V_1 > V_2$, $x_1^*(q)+x_2^*(q)$ *is decreasing in* q *and so the optimal tie-breaking rule is* $q = 0$.
- *If* $V_1 = V_2$, $x_1^*(q) + x_2^*(q)$ *is independent of* q.

First consider the case where the agents are symmetric so that $\beta = \frac{V_1}{V_2} = 1$. Observe that in this case, both agents exert equal effort, irrespective of the bias introduced by the tie-breaking rule q. Intuitively, this is because at any symmetric profile, a player's effort has zero marginal impact on the probability of a tie, and therefore, the tie-breaking rule q does not affect the symmetric equilibrium. With asymmetric agents, we again have that the stronger agent puts in greater effort, irrespective of the bias due to the tie-breaking rule q. In particular, the marginal impact of q on player i's effort is simply $V_i\beta p_0'(\beta)$. Since $\beta > 1$, $p_0'(\beta) < 0$ and it follows that the effort of both agents goes down as q increases. Thus, even if the designer had a more general objective function that was increasing in the effort of both agents, it would still be optimal for the designer to bias the tie-breaking rule completely in favor of the weaker player with $q = 0$. Formally, the result follows from the fact that the total equilibrium effort is linear in q with the coefficient $(V_1 + V_2)\beta p_0'(\beta)$. The full proof is in the appendix.

The Tullock contests, defined by $p_i(x_i, x_{-i}) = \frac{x_i^r}{x_i^r + x_{-i}^r}$ for $r \in (0, \infty)$, is an important class of ratio-form contest success functions that have been widely studied in the literature. There has been significant work in generalizing the Tullock contests to allow for the possibility of ties. We will consider three such generalizations. For two of these generalizations, the contest success functions take the ratio form and we discuss how are results apply to them next. The third generalization by Blavatskyy [7] is discussed later.

The first generalization we consider is given by $p_i(x_i, x_j) = \frac{x_i^{rk}}{(x_i^r + x_j^r)^k}$ with $k \geq 1$ so that the probability of ties is $p_0(x_i, x_j) = \frac{(x_i^r + x_j^r)^k - x_i^{rk} - x_j^{rk}}{(x_i^r + x_j^r)^k}$. This was proposed by Vesperoni and Yildizparlak [52]. In this case, we obtain the following lemma.

[8] Note that assumption 1 implies that the players payoffs Π_i are strictly concave in their actions x_i. Since the argument relies on there being a unique solution to the first-order conditions, we can relax assumption 1 so that payoffs are only required to be quasiconcave and the result would go through.

Lemma 1. *Suppose C is a ratio-form contest with $p_i(x_i, x_j) = \frac{x_i^{rk}}{(x_i^r + x_j^r)^k}$ where $k \geq 1$. If $rk \leq 1$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 rk \frac{\beta^{rk} + q(\beta^r - \beta^{rk})}{(1 + \beta^r)^{k+1}} \text{ and } x_2^* = V_2 rk \frac{\beta^{rk} + q(\beta^r - \beta^{rk})}{(1 + \beta^r)^{k+1}},$$

where $\beta = \frac{V_1}{V_2}$.

To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy assumptions 1, 2 when $rk \leq 1$ with $r \leq 1$, $k \geq 1$. Note that the conditions in the lemma are sufficient but not necessary for the result. The full proof is in the appendix.

The second generalization we consider is given by $p_i(x_i, x_j) = \frac{x_i^r}{x_i^r + kx_j^r}$ with $k \geq 1$ so that the probability of ties is $p_0(x_i, x_j) = 1 - \frac{x_i^r}{x_i^r + kx_j^r} - \frac{x_j^r}{kx_i^r + x_j^r}$. This was proposed by Jia [33]. In this case, we obtain the following lemma.

Lemma 2. *Suppose C is a ratio-form contest with $p_i(x_i, x_j) = \frac{x_i^r}{x_i^r + kx_j^r}$ where $k \geq 1$. If $r \leq 1$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 rk\beta^r \left(\frac{1}{(\beta^r + k)^2} + q \left(\frac{1}{(1 + k\beta^r)^2} - \frac{1}{(\beta^r + k)^2} \right) \right)$$

and

$$x_2^* = V_2 rk\beta^r \left(\frac{1}{(\beta^r + k)^2} + q \left(\frac{1}{(1 + k\beta^r)^2} - \frac{1}{(\beta^r + k)^2} \right) \right),$$

where $\beta = \frac{V_1}{V_2}$.

Again, we define the functions $p(\theta), p_0(\theta)$ and show that assumptions 1 and 2 are satisfied when $r \leq 1$. The full proof is in the appendix. Observe that in Lemmas 1 and 2, if we plug in $k = 1$, we get back the equilibrium characterization for the Tullock contests without ties (Nti [46]).

4 Difference-form contest success functions

In this section, we consider instances of our model where the distribution over the three outcomes depends only on the difference of the efforts $\theta = x_1 - x_2$ of the two players. Formally, we assume that there exist twice-differentiable functions $p : \mathbb{R} \to [0, 1]$ and $p_0 : \mathbb{R} \to [0, 1]$ such that

- $p_1(x_1, x_2) = p(\theta)$ and $p_2(x_1, x_2) = p(-\theta)$,
- $p_0(x_1, x_2) = p_0(\theta)$ and $p_0(\theta) = p_0(-\theta)$ for all $\theta \in \mathbb{R}$,
- $c(x) = x^2/2$.

We'll make the following assumption on these functions.

Assumption 3. Suppose $z_q(\theta) = p(\theta) + qp_0(\theta)$. For any $q \in [0,1]$, $z'_q(\theta) > 0$ and $z''_q(\theta) \in [-\frac{1}{V_1}, \frac{1}{V_1}]$.

Here again, $z_q(\theta)$ denotes the probability that player 1 eventually wins the prize and $1 - z_q(\theta)$ is the probability that player 2 wins eventually wins the prize. The assumption then says that a player's probability of winning is increasing in its own effort, decreasing in the effort of the other player. The second assumption ensures that the player's objective function is globally concave and a unique best response exists.

The next assumption we make is on the probability of ties $p_0(\theta)$ and is again motivated by the idea that the probability of a tie increases as the contest becomes more closely contested.

Assumption 4. The probability of tie $p_0(\theta)$ is increasing for $\theta \in (-\infty, 0]$ and decreasing for $\theta \in [0, \infty)$.

Let us now characterize the pure-strategy Nash equilibria of this difference-form contest game with ties.

Theorem 3. *Consider a difference-form contest satisfying assumption 3. The contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z'_q(\beta(q)) \text{ and } x_2^* = V_2 z'_q(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2) z'_q(\theta)$.

To prove the result, we show that there is a unique solution to the first-order conditions 1 and 2 for the contest game defined by x_1^* and x_2^*. The concavity of the payoffs implied by assumption 1 means that the second-order conditions for maximization are satisfied[9]. The full proof is in the appendix.

Note that the solution $\beta(q)$ to the equation $\theta = (V_1 - V_2) z'_q(\theta)$ is unique and ≥ 0 for all $q \in [0,1]$. It represents the difference in equilibrium effort levels exerted by the two agents. That is, the equilibrium effort levels x_1^*, x_2^* are such that $\frac{x_1^*}{x_2^*} = \frac{V_1}{V_2}$ and $x_1^* - x_2^* = \beta(q)$.

Next, we discuss the effect of the tie-breaking rule q on total equilibrium effort.

Theorem 4. *Consider a difference-form contest satisfying assumptions 3 and 4.*

- *If $V_1 > V_2$, $x_1^*(q) + x_2^*(q)$ is decreasing in q and so the optimal tie-breaking rule is $q = 0$.*
- *If $V_1 = V_2$, $x_1^*(q) + x_2^*(q)$ is independent of q.*

[9] Assumption 3 implies that the players payoffs Π_i are concave in x_i. The result goes through if the payoffs Π_i are instead assumed to be quasiconcave in x_i.

To prove Theorem 4, we can use Theorem 3 and our assumptions on p_0 to see how the total effort changes as we increase q. When $V_1 > V_2$, player 1 puts in greater effort so that $p_0'(x_1^* - x_2^*) < 0$. This implies that $R(q)$ is decreasing in q and thus, the optimal tie-breaking rule sets $q = 0$. In comparison, when $V_1 = V_2$, the agents exert equal effort irrespective of q and because $p_0'(0) = 0$, it follows that the total effort does not depend on the choice of q. The full proof is in the appendix.

The difference-form contest success functions were first studied in Hirshleifer [30]. A well-known example, with zero probability of ties, is the logit function $p_i(x_1, x_2) = \frac{\exp(x_i)}{\exp(x_i)+\exp(x_{-i})}$. While we are not aware of explicit work on generalizing these contest success functions to allow for the possibility of ties, we consider a couple of generalizations inspired from the literature on generalizing ratio-form contest success functions to allow for the possibility of ties.

The first generalization we consider is given by $p_i(x_i, x_j) = \left(\frac{\exp(x_i)}{\exp(x_i)+\exp(x_{-i})}\right)^k$ with $k \geq 1$. In this case, we obtain the following lemma.

Lemma 3. *Suppose C is a difference-form contest with $p_i(x_i, x_j) = \left(\frac{\exp(x_i)}{\exp(x_i)+\exp(x_{-i})}\right)^k$ with $k \geq 1$. There exist some $\bar{V} > 0$ such that if $\bar{V} \geq V_1 \geq V_2 > 0$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z_q'(\beta(q)) \text{ and } x_2^* = V_2 z_q'(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2) z_q'(\theta)$.

To prove the result, we show that $z_q'(\theta) > 0$ and also $\lim_{\theta\to\infty} z_q''(\theta) = 0$ and $\lim_{\theta\to-\infty} z_q''(\theta) = 0$ which implies $z_q''(\theta)$ is bounded between some M and m. Thus, if we take V_1 small enough so that $\frac{-1}{V_1} \leq m \leq M \leq \frac{1}{V_1}$, assumption 3 will be satisfied. We further show that this contest satisfies assumption 4 so that theorem 4 applies.

The next generalization we consider is given by $p_i(x_i, x_j) = \frac{\exp(x_i)}{\exp(x_i)+k\exp(x_{-i})}$ with $k \geq 1$. In this case, we obtain the following lemma.

Lemma 4. *Suppose C is a difference-form contest with $p_i(x_i, x_j) = \frac{\exp(x_i)}{\exp(x_i)+k\exp(x_{-i})}$ with $k \geq 1$. There exist some $\bar{V} > 0$ such that if $\bar{V} \geq V_1 \geq V_2 > 0$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z_q'(\beta(q)) \text{ and } x_2^* = V_2 z_q'(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2) z_q'(\theta)$.

As before, we show that $z_q'(\theta) > 0$ and also $\lim_{\theta\to\infty} z_q''(\theta) = 0$ and $\lim_{\theta\to-\infty} z_q''(\theta) = 0$ which implies $z_q''(\theta)$ is bounded between some M and m. It follows that 3 and 4 will be satisfied if V_1 is small enough and thus, we can apply theorems 3 and 4 to get the result.

5 Discussion

In this section, we consider some other examples of contests and also discuss the design of random tie-breaking rules.

5.1 Concave contests

Here we consider contests $C = (V_1, V_2, p_1(), p_2(), q, c())$ for which the contest success functions satisfy the following assumption.

Assumption 5. There exist impact functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ that are twice-differentiable, strictly increasing and concave so that

$$p_i(x_1, x_2) = \frac{f_i(x_i)}{f_1(x_1) + f_2(x_2) + 1} \text{ and } p_0(x_1, x_2) = \frac{1}{f_1(x_1) + f_2(x_2) + 1}.$$

This class of contest success functions with a possibility of tie was axiomatized by Blavatskyy [7]. For such contest success functions, it follows from Theorem 1 of Fu and Wu [25] that the contest has a unique pure-strategy Nash equilibrium.

Lemma 5. *Suppose the contest C satisfies assumption 5. Then, for any increasing and convex cost function c with $c(0) = 0$ and any tie-breaking rule $q \in [0, 1]$, the contest admits a unique pure-strategy Nash equilibrium.*

The proof proceeds by defining impact functions $g_1(x_1) = f_1(x_1)+q$ and $g_2(x_2) = f_2(x_2)+(1-q)$. Since these impact functions define a regular concave contest, it follows from Theorem 1 of Fu and Wu [25] that the game admits a unique pure-strategy Nash equilibrium[10].
The unique pure-strategy Nash equilibrium may involve both players exerting positive effort, in which case it is characterized by the first-order conditions 1 and 2. Going forward, we will focus on the special case where $f_1(x) = f_2(x) = f(x) = x^r$ for $r \in (0, 1]$ and $c(x) = x$. With these parametric assumptions, the first-order conditions are given by

$$V_1 P_2 r x_1^{r-1} = x_1^r + x_2^r + 1 = V_2 P_1 r x_2^{r-1},$$

where $P_2 = p_2 + (1 - q)p_0$ and $P_1 = p_1 + qp_0$.

We note here that for $r = 1$, the solution to these conditions is

$$x_1^* = \frac{V_1^2 V_2}{(V_1 + V_2)^2} - q \text{ and } x_2^* = \frac{V_1 V_2^2}{(V_1 + V_2)^2} - (1 - q),$$

and thus, the headstarts defined by the tie-breaking rule q act as a direct substitute for the effort exerted by them and the total equilibrium effort is independent of the choice of q. While we are not able to solve this case in full generality, we obtain the following for the special case of $r = 0.5$ and $V_1 = V_2$.

[10]Observe that a tie-breaking rule essentially defines additive head-starts q and $1 - q$ in the Blavatskyy [7] model. While previous work has studied the problem of finding optimal head-starts (Franke et al. [22], Fu and Wu [25], Li et al. [37]), our problem of finding the optimal tie-breaking rule is different in that the head-starts must add up to some constant.

Lemma 6. *Suppose the contest C is such that* $p_i(x_1, x_2) = \dfrac{\sqrt{x_i}}{\sqrt{x_i} + \sqrt{x_j} + 1}$, $c(x) = x$ *and* $V_1 = V_2 = V$.

- *If $q = 0.5$, the unique pure-strategy Nash equilibrium is given by*

$$x_1^* = x_2^* = x^* = \frac{1}{8}(V - \sqrt{2V+1} + 1).$$

- *If $q = 0$, the unique pure-strategy Nash equilibrium is given by*

$$x_1^* = \frac{V^2}{4(2V+1)} \text{ and } x_2^* = \frac{1}{4}\left(\frac{V+1}{\sqrt{2V+1}} - 1\right)^2.$$

- *The total equilibrium effort under $q = 0$ is greater than that under $q = 0.5$.*

Thus, even though the agents are identical with $V_1 = V_2$, a designer who cares about increasing effort is better off biasing the contest by setting $q = 0$ or $q = 1$. This is in contrast to Theorems 2 and 4 where we saw that for ratio-form and difference-form contest success functions, the choice of tie-breaking rule does not matter for equilibrium effort with symmetric agents. Lemma 6 motivates us to make the following conjecture for the general case of $r < 1$ and $V_1 \geq V_2$.

Conjecture 1. *Suppose C is a contest with* $p_i(x_1, x_2) = \dfrac{x_i^r}{x_i^r + x_j^r + 1}$ *for* $r \in (0, 1)$, $c(x) = x$ *and* $V_1 \geq V_2$.

- *If $V_1 > V_2$, the optimal tie-breaking rule is $q = 0$.*
- *If $V_1 = V_2$ the total effort is convex in $q \in [0, 1]$ with a minimum at $q = 0.5$ and maximum at $q = 0$ and $q = 1$.*

5.2 Random tie-breaking rules

In our model, we have assumed that the designer commits to a tie-breaking rule $q \in [0, 1]$. So $q = 0.5$ corresponds to the designer tossing a fair coin *after* the contest has ended in a tie to determine the winner. And $q = 0$ or $q = 1$ corresponds to the winner being pre-determined in case of a tie. As we have seen in Theorems 2, 4 and Lemma 6, there are many situations where an effort-maximizing designer would want to bias the contest in favor of one of the players. But it might not want to be perceived as biased towards any of the players. Even if the designer is not worried about being perceived as biased, it might not be ex-ante aware of the relative strengths of the two players. In such situations, the designer can choose a tie-breaking rule by tossing a fair coin *before* the contest begins to pre-determine the winner in case of a tie. This corresponds to a *random tie-breaking rule Q* which defines the tie-breaking rule $Q = 0$ with probability 0.5 and $Q = 1$ with probability 0.5. This random tie-breaking rule is unbiased ex-ante because $\mathbb{E}[Q] = 0.5$.

More generally, a random variable Q with support in $[0,1]$ defines a *random tie-breaking rule* and we say it is *unbiased* if $\mathbb{E}[Q] = 0.5$. The timing of the contest with random tie-breaking rules is as follows:

1. The contest designer chooses a random variable Q with support in $[0,1]$. This is the random tie-breaking rule.

2. The value of Q is revealed. Let's call it $q \in [0,1]$. This is the tie-breaking rule and is observed by all participants.

3. Agents decide their effort levels.

4. The contest outcome is revealed and in case of a tie, player 1 is chosen as winner with probability q and player 2 is chosen as winner with probability $1-q$.

We can now consider the problem of choosing a random tie-breaking rule Q to maximize expected total effort $\mathbb{E}[x_1(Q) + x_2(Q)]$ under the constraint that it is unbiased $\mathbb{E}[Q] = 0.5$. We will discuss implications of our results for the design of random unbiased tie-breaking rules.

First, for ratio-form contests, we obtain the following as a corollary of Theorem 1.

Corollary 2. *In a ratio-form contest satisfying assumptions 1 and 2 and $V_1 \geq V_2$, any random unbiased tie-breaking rule leads to the same expected total effort.*

This follows from the fact that the total equilibrium effort is linear in the tie-breaking rule q. More precisely, for any random tie-breaking rule Q with the constraint $\mathbb{E}[Q] = 0.5$, the expected total effort will be

$$\mathbb{E}[x_1(q) + x_2(q)] = (V_1 + V_2)\beta \left(p'(\beta) + \frac{1}{2} p_0'(\beta) \right),$$

where $\beta = \frac{V_1}{V_2}$.

For the difference-form contests we identify a sufficient condition under which breaking ties by tossing a fair coin before the contest would be optimal. Formally, we have the following:

Lemma 7. *Consider a difference-form contest satisfying assumptions 3 and 4. If $V_1 > V_2$ and $p_0''(\theta) < 0$ for all $\theta \in [0, \sqrt{2V_1}]$, the total effort is convex in q and the optimal unbiased random tie-breaking rule chooses $Q = 0$ and $Q = 1$ with equal probability.*

We note that when V_1 is small enough, the assumption $p_0''(\theta) < 0$ for all $\theta \in [0, \sqrt{2V_1}]$ is satisfied for the difference-form contest success functions considered in Lemmas 3 and 4. This suggests that a designer who cares about maximizing effort may be better off pre-determining the winner in case of a tie by tossing a fair coin before the contest as compared to the traditional practice of tossing a fair coin after the contest.

Now we will consider some simple contests and illustrate how the timing of the tie-breaker may influence the total effort in equilibrium.

5.2.1 Single-shot contest

In this subsection, we consider an instance of our model where

- $p_i(x_i, x_{-i}) = x_i(1 - x_{-i})$
- $c(x) = x^2$
- $2 \geq V_1 \geq V_2 > 0$

A motivation for this specification is a situation where the two players get a single shot at a basketball hoop and a player wins if and only if it is the only one to make the shot. The players exert effort cost x_i^2 to improve their individual probability x_i of making the shot. The resulting probabilities of winning the contest and the cost of effort will then be as described above. The next result identifies the optimal tie-breaking rule for this instance.

Lemma 8. *In a single-shot contest, the optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$). With symmetric agents, the optimal unbiased tie-breaking rule predetermines a winner in case of a tie by tossing a fair coin.*

To prove the result, we characterize the unique pure strategy Nash equilibrium of the single-shot contest game using equations 1 and 2 to get that total expected effort equals

$$R(b) = x_1^* + x_2^* = \frac{V_1(1-b) + V_2(1+b) + b^2 V_1 V_2}{4 + b^2 V_1 V_2}$$

where $b = 2q - 1$. We can then show that $R(b)$ is maximized at $b = -1$. With $V_1 = V_2$, $R(b)$ is maximized at both $b = -1, 1$ and so an unbiased tie-breaking rule would simply bias the contest in a fair way in favor of one the agents by tossing a fair coin before the contest. We believe this holds even in the case where $V_1 > V_2$ but haven't been able to prove it yet. But we can show that for any $V_1 > V_2$, breaking ties by tossing a fair coin before the contest leads to greater expected effort as compared to tossing a fair coin after the contest.

5.2.2 Sudden death contest

In this subsection, we consider an instance of our model where

- $p_i(x_i, x_{-i}) = \dfrac{x_i(1 - x_{-i})}{x_i + x_{-i} - x_i x_{-i}}$
- $c(x) = x$

- $2 \geq V_1 \geq V_2 > 0$

A motivation for this specification is a sudden death contest where players get multiple shots at a Basketball hoop and they can expend effort x_i to better their probability x_i of getting the shot. Player i wins if it is the first player to make the shot. The contest is drawn if both players make their first shot after missing the same number of times. The resulting probabilities of winning the contest and the cost of effort will then be as described above.

Lemma 9. *In a sudden death contest with symmetric agents $2 \geq V \geq \frac{3}{2}$, the expected effort from running a fair tie-breaker after the contest is higher than that from running it before the contest.*

To prove the result, we obtain conditions for an action profile to be a pure strategy Nash equilibrium of the sudden death contest game using equations 1 and 2. Using these conditions, we obtain the symmetric NE for the case where $q = \frac{1}{2}$. We find that the equilibrium effort for each agent takes the form $x^* = 1 - \sqrt{1 - \frac{V}{2}}$. While we don't obtain the equilibrium effort for $q = 0$ or $q = 1$, we use the conditions to obtain an upper bound on the total effort of 1. We then show that $2x^* \geq 1 \iff V \geq \frac{3}{2}$ which implies the result.

6 Conclusion

We study two-player contests with the possibility of ties under both ratio-form and difference-form contest success functions. In these contests, we study the effect different tie-breaking rules have on the effort exerted by the players. When players are heterogeneous, we find that the total effort decreases as the probability that ties are broken in favor of the stronger agent increases. Thus, an effort-maximizing designer would prefer to commit to breaking ties in favor of the weaker agent. The result lends further support to the encouraging effect of leveling the playing the field on effort in contests with heterogeneous agents.

With symmetric agents, we find that the equilibrium is generally symmetric and does not depend on the choice of tie-breaking rule in the case of ratio-form and difference-form contests. The problem is more interesting for concave contests in which we conjecture that an effort-maximizing designer would prefer to break ties by tossing a fair coin before the contest begins as compared to the standard practice of breaking ties after the contest ends. We believe that the study of tie-breaking rules for concave contests and contests with more than two agents provide interesting directions for future research.

References

[1] Alcalde, J. and M. Dahm (2010): "Rent seeking and rent dissipation: A neutrality result," *Journal of Public Economics*, 94, 1–7.

[2] Amegashie, J. A. (2006): "A contest success function with a tractable noise parameter," *Public Choice*, 135–144.

[3] Baik, K. H. (1998): "Difference-form contest success functions and effort levels in contests," *European Journal of Political Economy*, 14, 685–701. 2

[4] ——— (2004): "Two-player asymmetric contests with ratio-form contest success functions," *Economic Inquiry*, 42, 679–689. 2, 6

[5] Bastani, S., T. Giebe, and O. Gürtler (2022): "Simple equilibria in general contests," *Games and Economic Behavior*, 134, 264–280.

[6] Beviá, C. and L. C. Corchón (2015): "Relative difference contest success function," *Theory and Decision*, 78, 377–398. 2

[7] Blavatskyy, P. R. (2010): "Contest success function with the possibility of a draw: Axiomatization," *Journal of Mathematical Economics*, 46, 267–276. 2, 6, 10

[8] Chang, R., Y. Wang, and L. Lv (2023): "An analysis of group contests with the possibility of a draw," *Journal of Industrial and Management Optimization*, 19, 3952–3975. 3

[9] Che, Y.-K. and I. Gale (2000): "Difference-form contests and the robustness of all-pay auctions," *Games and Economic Behavior*, 30, 22–43. 2

[10] Che, Y.-K. and I. L. Gale (1998): "Caps on political lobbying," *The american economic review*, 88, 643–651.

[11] Chowdhury, S. M., P. Esteve-González, and A. Mukherjee (2023): "Heterogeneity, leveling the playing field, and affirmative action in contests," *Southern Economic Journal*, 89, 924–974. 2, 3

[12] Cohen, C. and A. Sela (2007): "Contests with ties," *The BE Journal of Theoretical Economics*, 7. 3

[13] CorchÃ³n, L. C., M. Serena, et al. (2016): "Properties of Contests," Tech. rep.

[14] Corchón, L. C. and M. Serena (2018): "Contest theory," in *Handbook of Game Theory and Industrial Organization, Volume II*, Edward Elgar Publishing, 125–146. 2

[15] Deng, S., X. Wang, and Z. Wu (2018): "Incentives in lottery contests with draws," *Economics Letters*, 163, 1–5. 2

[16] Drugov, M. and D. Ryvkin (2017): “Biased contests for symmetric players,” *Games and Economic Behavior*, 103, 116–144.

[17] ——— (2022): “Hunting for the discouragement effect in contests,” *Review of Economic Design*, 1–27.

[18] Eden, M. et al. (2006): “Optimal ties in contests,” Tech. rep. 3

[19] Epstein, G. S., Y. Mealem, and S. Nitzan (2013): “Lotteries vs. All-Pay Auctions in Fair and Biased Contests,” *Economics & Politics*, 25, 48–60.

[20] Ewerhart, C. (2017): “Revenue ranking of optimally biased contests: The case of two players,” *Economics Letters*, 157, 167–170. 2

[21] ——— (2021): “A typology of military conflict based on the Hirshleifer contest,” *University of Zurich, Department of Economics, Working Paper*. 2

[22] Franke, J., W. Leininger, and C. Wasser (2018): “Optimal favoritism in all-pay auctions and lottery contests,” *European Economic Review*, 104, 22–37. 3, 10

[23] Fu, Q., J. Lu, and Y. Pan (2015): “Team contests with multiple pairwise battles,” *American Economic Review*, 105, 2120–2140.

[24] Fu, Q., J. Lu, and J. Zhang (2016): “Disclosure policy in Tullock contests with asymmetric stochastic entry,” *Canadian Journal of Economics/Revue canadienne d’économique*, 49, 52–75.

[25] Fu, Q. and Z. Wu (2020): “On the optimal design of biased contests,” *Theoretical Economics*, 15, 1435–1470. 2, 3, 10, 27

[26] Garfinkel, M. R. and S. Skaperdas (2007): “Economics of conflict: An overview,” *Handbook of defense economics*, 2, 649–709. 2

[27] Gelder, A., D. Kovenock, and B. Roberson (2019): “All-pay auctions with ties,” *Economic Theory*, 1–49. 3

[28] Gelder, A., D. Kovenock, and R. M. Sheremeta (2015): “Behavior in all-pay auctions with ties,” *Available at SSRN 2683568*.

[29] Haggiag, C., S. Oren, and E. Segev (2022): “Picking the Right Winner: Why Tie-Breaking in Crowdsourcing Contests Matters,” in *IJCAI-International Joint Conferences on Artificial Intelligence Organization*, 307–313.

[30] Hirshleifer, J. (1989): “Conflict and rent-seeking success functions: Ratio vs. difference models of relative success,” *Public choice*, 63, 101–112. 2, 9

[31] Imhof, L. and M. Kräkel (2014): “Tournaments with gaps,” *Economics Letters*, 122, 211–214. 3

[32] ——— (2016): "Ex post unbalanced tournaments," *The RAND Journal of Economics*, 47, 73–98. 3

[33] Jia, H. (2012): "Contests with the probability of a draw: a stochastic foundation," *Economic record*, 88, 391–406. 2, 3, 7

[34] Jia, H., S. Skaperdas, and S. Vaidya (2013): "Contest functions: Theoretical foundations and issues in estimation," *International Journal of Industrial Organization*, 31, 211–222. 2

[35] Klumpp, T. and M. K. Polborn (2006): "Primaries and the New Hampshire effect," *Journal of Public Economics*, 90, 1073–1114.

[36] Lazear, E. P. and S. Rosen (1981): "Rank-order tournaments as optimum labor contracts," *Journal of political Economy*, 89, 841–864.

[37] Li, B., Z. Wu, and Z. Xing (2023): "Optimally biased contests with draws," *Economics Letters*, 111076. 2, 3, 10

[38] Loury, G. C. (1979): "Market structure and innovation," *The quarterly journal of economics*, 93, 395–410. 2

[39] Malcomson, J. M. (1984): "Work incentives, hierarchy, and internal labor markets," *Journal of political economy*, 92, 486–507.

[40] ——— (1986): "Rank-order contracts for a principal with many agents," *The Review of Economic Studies*, 53, 807–817.

[41] Malueg, D. A. and A. J. Yates (2006): "Equilibria in rent-seeking contests with homogeneous success functions," *Economic Theory*, 719–727.

[42] Mealem, Y. and S. Nitzan (2016): "Discrimination in contests: a survey," *Review of Economic Design*, 20, 145–172. 2, 3

[43] Minchuk, Y. (2022): "Winner-pay contests with a no-winner possibility," *Managerial and Decision Economics*, 43, 1874–1879. 2

[44] Nalebuff, B. J. and J. E. Stiglitz (1983): "Prizes and incentives: towards a general theory of compensation and competition," *The Bell Journal of Economics*, 21–43. 3

[45] Nti, K. O. (1997): "Comparative statics of contests and rent-seeking games," *International Economic Review*, 43–59. 2

[46] ——— (1999): "Rent-seeking with asymmetric valuations," *Public Choice*, 98, 415–430. 2, 7

[47] ——— (2004): "Maximum efforts in contests with asymmetric valuations," *European journal of political economy*, 20, 1059–1066. 2

[48] Ridlon, R. and J. Shin (2013): "Favoring the winner or loser in repeated contests," *Marketing Science*, 32, 768–785.

[49] Skaperdas, S. (1996): "Contest success functions," *Economic theory*, 7, 283–290. 2

[50] Szech, N. (2015): "Tie-breaks and bid-caps in all-pay auctions," *Games and Economic Behavior*, 92, 138–149. 3

[51] Tullock, G. (2001): "Efficient rent seeking," *Efficient rent-seeking: Chronicle of an intellectual quagmire*, 3–16. 2

[52] Vesperoni, A. and A. Yildizparlak (2019): "Contests with draws: Axiomatization and equilibrium," *Economic inquiry*, 57, 1597–1616. 2, 3, 6

[53] Wang, Z. (2010): "The optimal accuracy level in asymmetric contests," *The BE Journal of Theoretical Economics*, 10. 2

[54] Yildizparlak, A. (2013): "A Contest Success Function Allowing for Ties with an Application to Soccer," *Available at SSRN 2270351*.

[55] ——— (2018): "An application of contest success functions for draws on European soccer," *Journal of sports economics*, 19, 1191–1212.

[56] Zhu, F. (2021): "On optimal favoritism in all-pay contests," *Journal of Mathematical Economics*, 95, 102472.

A Proofs for Section 3 (Ratio-form contest success functions)

Theorem 1. *Consider a ratio-form contest satisfying assumption 1. The contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 \beta z_q'(\beta) \text{ and } x_2^* = V_2 \beta z_q'(\beta),$$

where $\beta = \frac{V_1}{V_2} \geq 1$.

Proof. In a ratio-form contest $C = (V_1, V_2, p_1(), p_2(), q, c())$, there exist functions p and p_0 so that we can rewrite players payoff under profile x_1, x_2 as

$$\Pi_1 = V_1 z_q(\theta) - x_1 \text{ and } \Pi_2 = V_2(1 - z_q(\theta)) - x_2$$

where $\theta = \dfrac{x_1}{x_2}$ and $z_q(\theta) = p(\theta) + qp_0(\theta)$.

Since the prize is valued positively by both players and the probability of winning the contest with $x_i = 0, x_j > 0$ is 0 for player i and 1 for player j, any pure-strategy Nash equilibrium involves positive effort levels by both players.

Now, observe that

$$\frac{\partial \Pi_1}{\partial x_1} = \frac{V_1}{x_2} z_q'(\theta) - 1 \text{ and } \frac{\partial \Pi_2}{\partial x_2} = \frac{x_1 V_2}{x_2^2} z_q'(\theta) - 1$$

and

$$\frac{\partial^2 \Pi_1}{\partial x_1^2} = \frac{V_1}{x_2^2}(z_q''(\theta)) < 0 \text{ and } \frac{\partial^2 \Pi_2}{\partial x_2^2} = -\frac{x_1 V_2}{x_2^3}\left(2z_q'(\theta) + \theta z_q''(\theta)\right) < 0.$$

From assumption 1, Π_i is concave in x_i for any x_j. Since any pure-strategy Nash equilibrium must satisfy the first-order conditions

$$V_1 z_q'(\theta) = x_2 \text{ and } x_1 V_2 z_q'(\theta) = x_2^2,$$

concavity of payoffs implies that the second-order conditions are satisfied at any solution to the first-order conditions. Observe that any solution to the two first-order conditions must satisfy

$$\frac{x_1}{x_2} = \frac{V_1}{V_2} = \beta.$$

But this condition uniquely pins down the solution to the first-order conditions:

$$x_1^* = V_1 \beta z_q'(\beta) \text{ and } x_2^* = V_2 \beta z_q'(\beta).$$

□

Theorem 2. *Consider a ratio-form contest satisfying assumptions 1 and 2.*

- *If $V_1 > V_2$, $x_1^*(q) + x_2^*(q)$ is decreasing in q and so the optimal tie-breaking rule is $q = 0$.*
- *If $V_1 = V_2$, $x_1^*(q) + x_2^*(q)$ is independent of q.*

Proof. From Theorem 1, the total effort in equilibrium is given by

$$R(q) = (V_1 + V_2)\beta(p'(\beta) + qp_0'(\beta)).$$

Observe that the total effort is linear in q with the coefficient $(V_1 + V_2)\beta p_0'(\beta)$ and the sign of the coefficient is determined by $p_0'(\beta)$. When $\beta = \frac{V_1}{V_2} > 1$ assumption 2 implies that $p_0'(\beta) < 0$ and it follows that the the total effort is decreasing in q. So the optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$).

Similarly, when $\beta = \frac{V_1}{V_2} = 1$ assumption 2 implies that $p_0'(\beta) = 0$ and it follows that the equilibrium effort, and thus, the total effort, is independent of q. So the choice of the tie-breaking rule does not matter in this case.

□

Lemma 1. *Suppose C is a ratio-form contest with $p_i(x_i, x_j) = \frac{x_i^{rk}}{(x_i^r + x_j^r)^k}$ where $k \geq 1$. If $rk \leq 1$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 rk \frac{\beta^{rk} + q(\beta^r - \beta^{rk})}{(1 + \beta^r)^{k+1}} \text{ and } x_2^* = V_2 rk \frac{\beta^{rk} + q(\beta^r - \beta^{rk})}{(1 + \beta^r)^{k+1}},$$

where $\beta = \frac{V_1}{V_2}$.

Proof. To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy assumptions 1, 2 when $rk \leq 1$ with $r \leq 1$, $k \geq 1$.

For the given contest, we can define the functions

$$p(\theta) = \frac{\theta^{rk}}{(1 + \theta^r)^k}, \quad p_0(\theta) = \frac{(1 + \theta^r)^k - 1 - \theta^{rk}}{(1 + \theta^r)^k}$$

so that

$$z_q(\theta) = \frac{\theta^{rk} + q((1 + \theta^r)^k - 1 - \theta^{rk})}{(1 + \theta^r)^k}.$$

Let us first identify conditions under which assumption 1 is satisfied.

To do so, we have that

$$\begin{aligned} z_q'(\theta) &= \frac{(1 + \theta^r)^k(rk\theta^{rk-1}(1 - q) + qkr(1 + \theta^r)^{k-1}\theta^{r-1}) - (\theta^{rk} + q((1 + \theta^r)^k - 1 - \theta^{rk}))(kr(1 + \theta^r)^{k-1}\theta^{r-1})}{(1 + \theta^r)^{2k}} \\ &= kr\theta^{r-1}\frac{(\theta^{rk-r}(1 - q)) + q}{(1 + \theta^r)^{k+1}} \\ &= kr\frac{\theta^{rk-1}(1 - q) + q\theta^{r-1}}{(1 + \theta^r)^{k+1}}. \end{aligned}$$

Also,

$$z_q''(\theta) = kr\theta^{r-2}\frac{q(r-1)+\theta^{rk-r}(rk-1)(1-q)-\theta^{rk}(1+r)(1-q)-\theta^r q(1+kr)}{(1+\theta^r)^{k+2}}$$
$$= kr\theta^{r-2}\frac{q(r-1-\theta^r(1+kr))+(1-q)\theta^{rk-r}(rk-1-\theta^r(1+r))}{(1+\theta^r)^{k+2}}.$$

Lastly,

$$2z_q'(\theta)+\theta z_q''(\theta) = kr\theta^{r-1}\frac{q(r+1+\theta^r(1-kr))+(1-q)\theta^{rk-r}(rk+1+\theta^r(1-r))}{(1+\theta^r)^{k+2}}.$$

From the above expressions, we have that $z_q'(\theta) > 0$ for all $\theta > 0$. The condition $rk \leq 1$ with $r \leq 1$ and $k \geq 1$ is sufficient for both $z_q''(\theta) < 0$ and $2z_q'(\theta)+\theta z_q''(\theta) > 0$. Thus, we have that if $rk \leq 1$, assumption 1 is satisfied.

Observe that

$$p_0'(\theta) = kr\frac{\theta^r - \theta^{rk}}{\theta(1+\theta^r)^{k+1}},$$

which is ≥ 0 for $\theta \leq 1$ and ≤ 0 for $\theta \geq 1$ since $k \geq 1$. Thus, assumption 2 is also satisfied.

□

Lemma 2. *Suppose C is a ratio-form contest with $p_i(x_i, x_j) = \frac{x_i^r}{x_i^r + kx_j^r}$ where $k \geq 1$. If $r \leq 1$, then for any $q \in [0,1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 rk\beta^r\left(\frac{1}{(\beta^r+k)^2} + q\left(\frac{1}{(1+k\beta^r)^2} - \frac{1}{(\beta^r+k)^2}\right)\right)$$

and

$$x_2^* = V_2 rk\beta^r\left(\frac{1}{(\beta^r+k)^2} + q\left(\frac{1}{(1+k\beta^r)^2} - \frac{1}{(\beta^r+k)^2}\right)\right),$$

where $\beta = \frac{V_1}{V_2}$.

Proof. To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy assumptions 1, 2 when $r \leq 1$. We further show that the condition in Theorem 2 is also satisfied when $r \leq 1$ which implies the result.

For the given contest, we can define the functions

$$p(\theta) = \frac{\theta^r}{\theta^r + k}, \quad p_0(\theta) = 1 - \frac{\theta^r}{\theta^r+k} - \frac{1}{k\theta^r+1}$$

so that

$$z_q(\theta) = \frac{\theta^r(1+k\theta^r+q(k^2-1))}{(1+k\theta^r)(\theta^r+k)}.$$

Let us first identify conditions under which assumption 1 is satisfied.
To do so, we have that

$$z_q'(\theta) = kr\theta^{r-1}\frac{(1-q)(1+k\theta^r)^2 + q(\theta^r+k)^2}{(1+k\theta^r)^2(\theta^r+k)^2}.$$

Also,

$$z_q''(\theta) = -kr\theta^{r-2}\left(2r\theta^r\left(\frac{1-q}{(\theta^r+k)^3} + \frac{qk}{(1+k\theta^2)^3}\right) + (1-r)\left(\frac{1-q}{(\theta^r+k)^2} + \frac{q}{(1+k\theta^2)^2}\right)\right).$$

Lastly,

$$2z_q'(\theta) + \theta z_q''(\theta) = kr\theta^{r-1}\left(\frac{(1-q)}{(\theta^r+k)^2}\left(1+r-\frac{2r\theta^r}{\theta^r+k}\right) + \frac{q}{(1+k\theta^r)^2}\left(1+r-\frac{2rk\theta^r}{1+k\theta^r}\right)\right).$$

From the above expressions, we have that $z_q'(\theta) > 0$ for all $\theta > 0$. The condition $r \leq 1$ is sufficient for both $z_q''(\theta) < 0$ and $2z_q'(\theta) + \theta z_q''(\theta) > 0$. Thus, we have that if $r \leq 1$, assumption 1 is satisfied.

Observe that

$$\begin{aligned}
p_0'(\theta) &= kr\theta^{r-1}\frac{(\theta^r+k)^2 - (1+k\theta^r)^2}{(1+k\theta^r)^2(\theta^r+k)^2} \\
&= kr\theta^{r-1}\frac{(\theta^r+k+1+k\theta^r)(\theta^r+k-1-k\theta^r)}{(1+k\theta^r)^2(\theta^r+k)^2} \\
&= kr\theta^{r-1}\frac{(\theta^r+k+1+k\theta^r)(k-1)(1-\theta^r)}{(1+k\theta^r)^2(\theta^r+k)^2},
\end{aligned}$$

which is ≥ 0 for $\theta \leq 1$ and ≤ 0 for $\theta \geq 1$ since $k \geq 1$. Thus, assumption 2 is also satisfied.

□

B Proofs for Section 4 (Difference-form contest success functions)

Theorem 3. *Consider a difference-form contest satisfying assumption 3. The contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z_q'(\beta(q)) \text{ and } x_2^* = V_2 z_q'(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)z_q'(\theta)$.

Proof. In a difference-form contest $C = (V_1, V_2, p_1(), p_2(), q, c())$, there exist functions p and p_0 so that we can rewrite players payoff under profile x_1, x_2 as

$$\Pi_1 = V_1 z_q(\theta) - \frac{x_1^2}{2} \text{ and } \Pi_2 = V_2(1 - z_q(\theta)) - \frac{x_2^2}{2}$$

where $\theta = x_1 - x_2$ and $z_q(\theta) = p(\theta) + qp_0(\theta)$.

Now, observe that

$$\frac{\partial \Pi_1}{\partial x_1} = V_1 z_q'(\theta) - x_1 \text{ and } \frac{\partial \Pi_2}{\partial x_2} = V_2 z_q'(\theta) - x_2$$

and

$$\frac{\partial^2 \Pi_1}{\partial x_1^2} = V_1(z_q''(\theta)) - 1 < 0 \text{ and } \frac{\partial^2 \Pi_2}{\partial x_2^2} = -V_2(z_q''(\theta)) - 1 < 0.$$

From assumption 3, Π_i is concave in x_i for any x_j. Since any pure-strategy Nash equilibrium must satisfy the first-order conditions

$$V_1 z_q'(\theta) = x_1 \text{ and } V_2 z_q'(\theta) = x_2,$$

concavity of payoffs implies that the second-order conditions for maximization are satisfied at any solution to the first-order conditions. Now any solution to the two first-order conditions must satisfy the following:

$$\frac{x_1}{x_2} = \frac{V_1}{V_2}$$

and

$$x_1 - x_2 = (V_1 - V_2) z_q'(x_1 - x_2).$$

Consider the equation $\theta - (V_1 - V_2) z_q'(\theta) = 0$. The derivative of the left hand side is $1 - (V_1 - V_2) z_q''(\theta)$ which is ≥ 0 as long as $z_q''(\theta) \leq \frac{1}{V_1 - V_2}$. From assumption 3, $z_q''(\theta) \leq \frac{1}{V_1}$ which implies $z_q''(\theta) \leq \frac{1}{V_1 - V_2}$. Thus, the left hand side is monotone increasing in θ. Also observe that at $\theta = 0$, the left hand side is $-(V_1 - V_2) z_q'(0) < 0$. Thus, there is a unique solution to the equation $\theta - (V_1 - V_2) z_q'(\theta) = 0$. Let $\beta(q)$ denote this unique solution to the equation $\theta - (V_1 - V_2) z_q'(\theta) = 0$. The uniqueness of $\beta(q)$ implies that there is a unique solution to the first-order conditions and it takes the form

$$x_1^* = \frac{\beta(q) V_1}{V_1 - V_2} \text{ and } x_2^* = \frac{\beta(q) V_2}{V_1 - V_2}.$$

Equivalently, we can write it as

$$x_1^* = V_1 z_q'(\beta(q)) \text{ and } x_2^* = V_2 z_q'(\beta(q)).$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2) z_q'(\theta)$. □

Theorem 4. *Consider a difference-form contest satisfying assumptions 3 and 4.*

- *If* $V_1 > V_2$, $x_1^*(q)+x_2^*(q)$ *is decreasing in* q *and so the optimal tie-breaking rule is* $q = 0$.
- *If* $V_1 = V_2$, $x_1^*(q) + x_2^*(q)$ *is independent of* q.

Proof. From Theorem 3, the total effort in the unique pure-strategy Nash equilibrium is given by

$$R(q) = (V_1 + V_2)z_q'(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)z_q'(\theta)$.

Consider the case where $V_1 > V_2$. In this case, we have that

$$R(q) = (V_1 + V_2)z_q'(\beta(q)) = \frac{V_1 + V_2}{V_1 - V_2}\beta(q)$$

from the equation defining $\beta(q)$. Thus, $R'(q) = \dfrac{V_1 + V_2}{V_1 - V_2}\beta'(q)$ which implies that the total equilibrium effort $R(q)$ goes in the same direction as the difference in equilibrium effort $\beta(q)$ as we change q. From the characterizing equation, we know that

$$\beta'(q) = (V_1 - V_2)\left(p''(\beta(q))\beta'(q) + p_0'(\beta(q)) + qp_0''(\beta(q))\beta'(q)\right),$$

which implies

$$\beta'(q)\left(1 - (V_1 - V_2)\left(p''(\beta(q)) + qp_0''(\beta(q))\right)\right) = (V_1 - V_2)p_0'(\beta(q)).$$

Given that $z_q''(\theta) \in [-\frac{1}{V_1}, \frac{1}{V_1}]$ from assumption 3, we know that $\beta'(q)$ has the same sign as $p_0'(\beta(q))$. We already know $\beta(q) > 0$ and therefore, from assumption 4, we get that $\beta'(q) < 0$. Therefore, the total equilibrium effort $R(q)$ is decreasing in q and it follows that the optimal tie breaking rule breaks ties in favor of the weaker agent by setting $q = 0$.

When $V_1 = V_2$, $\beta(q) = 0$ for all $q \in [0, 1]$ and thus, $R(q) = (V_1 + V_2)(p'(0) + qp_0'(0))$. But we know from assumption 4 that $p_0'(0) = 0$ and thus, we get that the total effort equals $R(q) = (V_1 + V_2)p'(0)$ for any $q \in [0, 1]$. Thus, with symmetric agents, the choice of the tie-breaking rule does not matter for equilibrium, and thus, total effort. □

Lemma 3. *Suppose* C *is a difference-form contest with* $p_i(x_i, x_j) = \left(\frac{\exp(x_i)}{\exp(x_i)+\exp(x_{-i})}\right)^k$ *with* $k \geq 1$. *There exist some* $\bar{V} > 0$ *such that if* $\bar{V} \geq V_1 \geq V_2 > 0$, *then for any* $q \in [0, 1]$, *the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z_q'(\beta(q)) \text{ and } x_2^* = V_2 z_q'(\beta(q)),$$

where $\beta(q)$ *is the unique solution to the equation* $\theta = (V_1 - V_2)z_q'(\theta)$.

Proof. To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy assumptions 3, 4 if V_1 is small enough. The result then follows from Theorems 3 and 4.

For the given contest, we can define the functions

$$p(\theta) = \frac{e^{k\theta}}{(1+e^\theta)^k}, \quad p_0(\theta) = \frac{(1+e^\theta)^k - 1 - e^{k\theta}}{(1+e^\theta)^k}$$

so that

$$z_q(\theta) = (1-q)\frac{e^{k\theta}}{(1+e^\theta)^k} + q\left(1 - \frac{1}{(1+e^\theta)^k}\right).$$

Let us first identify conditions under which assumption 3 is satisfied.

To do so, we have that

$$z_q'(\theta) = \frac{ke^\theta}{(1+e^\theta)^{k+1}}\left((1-q)e^{(k-1)\theta} + q\right).$$

Also,

$$z_q''(\theta) = \frac{ke^\theta}{(1+e^\theta)^{k+2}}\left((1-q)e^{(k-1)\theta}(k-e^\theta) + q(1-ke^\theta)\right).$$

Observe that $z_q'(\theta) > 0$. Letting $t = e^\theta$, we get

$$z_q''(\theta) = \frac{kt}{(1+t)^{k+2}}\left((1-q)t^{(k-1)}(k-t) + q(1-kt)\right).$$

Since $\lim_{t\to\infty} z_q''(\theta) = 0$ and $\lim_{t\to 0} z_q''(\theta) = 0$, there exists some bounds m, M such that $m \le z_q''(\theta) \le M$ for all $\theta \in \mathbb{R}$. Thus, if we take V_1 small enough so that $\frac{-1}{V_1} \le m \le M \le \frac{1}{V_1}$, assumption 3 will be satisfied.

We also note that this contest success function satisfies assumption 4 as

$$p_0'(\theta) = \frac{ke^\theta}{(1+e^\theta)^{k+1}}(1 - e^{(k-1)\theta}),$$

which is > 0 for $\theta < 0$ and < 0 for $\theta > 0$. Thus, theorem 4 applies. □

Lemma 4. *Suppose C is a difference-form contest with $p_i(x_i, x_j) = \frac{\exp(x_i)}{\exp(x_i)+k\exp(x_{-i})}$ with $k \ge 1$. There exist some $\bar{V} > 0$ such that if $\bar{V} \ge V_1 \ge V_2 > 0$, then for any $q \in [0,1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z_q'(\beta(q)) \text{ and } x_2^* = V_2 z_q'(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)z_q'(\theta)$.

Proof. To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy assumptions 3, 4 if V_1 is small enough. The result then follows from Theorems 3 and 4.

For the given contest, we can define the functions

$$p(\theta) = \frac{e^\theta}{k + e^\theta}, \quad p_0(\theta) = 1 - \frac{e^\theta}{k + e^\theta} - \frac{1}{ke^\theta + 1}$$

so that

$$z_q(\theta) = (1-q)\frac{e^\theta}{k + e^\theta} + q\frac{ke^\theta}{1 + ke^\theta}.$$

Let us first identify conditions under which assumption 3 is satisfied.

To do so, we have that

$$z'_q(\theta) = ke^\theta \left(\frac{1-q}{(k+e^\theta)^2} + \frac{q}{(1+ke^\theta)^2} \right).$$

Also,

$$z''_q(\theta) = ke^\theta \left(\frac{(1-q)(k^2 - e^{2\theta})}{(k+e^\theta)^4} + \frac{q(1-k^2e^{2\theta})}{(1+ke^\theta)^4} \right).$$

Again, we have that $z'_q(\theta) > 0$ and also $\lim_{\theta \to \infty} z''_q(\theta) = 0$ and $\lim_{\theta \to -\infty} z''_q(\theta) = 0$ which implies $z''_q(\theta)$ is bounded between some M and m. Thus, if we take V_1 small enough so that $\frac{-1}{V_1} \leq m \leq M \leq \frac{1}{V_1}$, assumption 3 will be satisfied.

We also note that this contest success function satisfies assumption 4 as

$$p'_0(\theta) = ke^\theta \left(\frac{1}{(1+ke^\theta)^2} - \frac{1}{(k+e^\theta)^2} \right),$$

which is > 0 for $\theta < 0$ and < 0 for $\theta > 0$. Thus, theorem 4 applies. □

C Proofs for Section 5 (Discussion)

Lemma 5. *Suppose the contest C satisfies assumption 5. Then, for any increasing and convex cost function c with $c(0) = 0$ and any tie-breaking rule $q \in [0,1]$, the contest admits a unique pure-strategy Nash equilibrium.*

Proof. Let f_1, f_2 denote the increasing and concave impact functions that define the contest success functions p_1, p_2, p_0. For any $q \in [0,1]$, we can write the probability that player i eventually wins the contest as

$$P_i = \frac{g_i(x_i)}{g_1(x_1) + g_2(x_2)},$$

where

$$g_1(x_1) = f_1(x_1) + qs \text{ and } g_2(x_2) = f_2(x_2) + (1-q)s.$$

Observe that g_i is also twice-differentiable, strictly increasing and concave. Thus, it follows from Theorem 1 of Fu and Wu [25] that there exists a unique pure-strategy Nash equilibrium in this game. □

Lemma 6. *Suppose the contest C is such that* $p_i(x_1, x_2) = \dfrac{\sqrt{x_i}}{\sqrt{x_i} + \sqrt{x_j} + 1}$, $c(x) = x$ *and* $V_1 = V_2 = V$.

- *If $q = 0.5$, the unique pure-strategy Nash equilibrium is given by*

$$x_1^* = x_2^* = x^* = \frac{1}{8}(V - \sqrt{2V+1} + 1).$$

- *If $q = 0$, the unique pure-strategy Nash equilibrium is given by*

$$x_1^* = \frac{V^2}{4(2V+1)} \text{ and } x_2^* = \frac{1}{4}\left(\frac{V+1}{\sqrt{2V+1}} - 1\right)^2.$$

- *The total equilibrium effort under $q = 0$ is greater than that under $q = 0.5$.*

Proof. Note that for general $r < 1$ and $V_1 > V_2$, the first-order conditions imply that

$$\begin{aligned} & V_1 P_2 r x_1^{r-1} = V_2 P_1 r x_2^{r-1} \\ \iff & \left(\frac{V_2(x_1^r + q)}{V_1(x_2^r + 1 - q)}\right)^{\frac{1}{1-r}} = \frac{x_2}{x_1}. \end{aligned}$$

Using this equation, we get that

- for $q = 0$, $x_1 = \frac{V_1}{V_2}(x_2 + x_2^{1-r})$,
- for $q = 1$, we have $x_2 = \frac{V_2}{V_1}(x_1 + x_1^{1-r})$,
- for $q = 0.5$ and $V_1 = V_2$, we have $x_1 = x_2 = x^*$ and in particular, x^* solves $4x^r - Vrx^{r-1} + 2 = 0$.

Thus, for $q = 0.5$ with $r = 0.5$ and $V_1 = V_2 = V$, we can solve the equation to get that the unique NE is given by

$$x_1^* = x_2^* = x^* = \frac{1}{8}(V - \sqrt{2V+1} + 1).$$

Similarly, with $r = 0.5$ and $V_1 = V_2 = V$, we can solve the first-order conditions for $q = 0$ (or $q = 1$) and get that the unique NE is given by

$$x_1^* = \frac{V^2}{4(2V+1)} \text{ and } x_2^* = \frac{1}{4}\left(\frac{V+1}{\sqrt{2V+1}} - 1\right)^2.$$

For the last claim, observe that

$$
\begin{aligned}
x_1^*(0.5) + x_2^*(0.5) - x_1^*(0) - x_2^*(0) &= \frac{1}{4}\left(V - \sqrt{2V+1} + 1 - \frac{V^2}{2V+1} - \frac{(V+1-\sqrt{2V+1})^2}{2V+1}\right) \\
&= \frac{1}{4}\left(V + 1 - \frac{1}{2V+1}\left(2V^2 + 4V + 2 - \sqrt{2V+1}\right)\right) \\
&= \frac{1}{4}\left(V + \frac{1}{\sqrt{2V+1}} - 1 - \frac{2V^2}{2V+1}\right) \\
&= \frac{\sqrt{2V+1} - V - 1}{4(2V+1)} \\
&< 0.
\end{aligned}
$$

□

Lemma 7. *Consider a difference-form contest satisfying assumptions 3 and 4. If $V_1 > V_2$ and $p_0''(\theta) < 0$ for all $\theta \in [0, \sqrt{2V_1}]$, the total effort is convex in q and the optimal unbiased random tie-breaking rule chooses $Q = 0$ and $Q = 1$ with equal probability.*

Proof. From Theorem 3, the total effort in the unique pure-strategy Nash equilibrium is given by

$$R(q) = (V_1 + V_2) z_q'(\beta(q)) = \frac{V_1 + V_2}{V_1 - V_2}\beta(q),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2) z_q'(\theta)$.

Thus,

$$R''(q) = (V_1 + V_2) z_q'(\beta(q)) = \frac{V_1 + V_2}{V_1 - V_2}\beta''(q).$$

Note that the solution $\beta(q)$ of an implicit equation $F(\theta, q) = 0$ is convex iff

$$\frac{\partial F}{\partial q}\frac{\partial^2 F}{\partial \theta \partial q} - \frac{\partial F}{\partial \theta}\frac{\partial^2 F}{\partial q^2} \geq 0.$$

In our case, $\beta(q)$ is the solution to the equation $(V_1 - V_2)\left(p'(\theta) + q p_0'(\theta)\right) - \theta = 0$.

Then, $\beta(q)$ is convex if and only if

$$(V_1 - V_2)^2 p_0'(\theta) p_0''(\theta) \geq 0.$$

We know that for $V_1 > V_2$, $\theta > 0$ and so $p_0'(\theta) < 0$. Therefore, we need $p_0''(\theta) < 0$ for $\beta(q)$, and thus, $R(q)$ to be convex. Observe also that player 1's equilibrium effort x_1^* must be such that $V_1 - \frac{x_1^2}{2} \geq 0 \iff x_1^* \leq \sqrt{2V_1}$. Since $x_2^* \geq 0$, it follows that $\beta(q) \leq \sqrt{2V_1}$ for any q. Thus, if we have $p_0''(\theta) \leq 0$ for $\theta \in [0, \sqrt{2V_1}]$, we have that the total effort is convex in q. It follows from convexity then that the optimal random tie-breaking rule in the class of unbiased rules breaks ties in fair manner before the contest. □

Lemma 8. *In a single-shot contest, the optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$). With symmetric agents, the optimal unbiased tie-breaking rule predetermines a winner in case of a tie by tossing a fair coin.*

Proof. In the single shot contest game, we have $\frac{\partial p_i}{\partial x_i} = (1 - x_{-i})$, $\frac{\partial p_0}{\partial x_i} = 2x_{-i} - 1$ and $c'(x) = 2x$. Plugging these into the two first order conditions in equations 1 and 2 and letting $q = \frac{1+b}{2}$ with bias $b \in [-1, 1]$, we get that the NE satisfies:

$$V_1(1 - q + x_2 b) = 2x_1$$
$$V_2(q - x_1 b) = 2x_2$$

Thus, we have two linear equations in two variables which we can solve to get the Nash equilibrium

$$x_1^* = \frac{V_1\left(1 - b + bqV_2\right)}{4 + b^2 V_1 V_2} \quad x_2^* = \frac{V_2\left(1 + b - b(1-q)V_1\right)}{4 + b^2 V_1 V_2}$$

so that

$$R(b) = x_1^* + x_2^* = \frac{V_1(1-b) + V_2(1+b) + b^2 V_1 V_2}{4 + b^2 V_1 V_2}$$

Since $V_1 \geq V_2$, we have that for any $b > 0$, $R(b) \leq R(-b)$. And therefore, the optimal bias $b \in [-1, 0]$. In this domain, we can differentiate the objective with respect to b and show that $R(b)$ is decreasing in b. We have

$$R'(b) = \frac{(4 - b^2 V_1 V_2)(V_2 - V_1) - 2bV_1V_2(V_1 + V_2 - 4)}{(4 + b^2 V_1 V_2)^2}$$

Since $2 \geq V_1 \geq V_2 > 0$, $R'(b) < 0$ for $b \in [-1, 0]$ which implies $R(b)$ is decreasing in this domain. Thus, the solution to the designer's problem when it is unconstrained is to set $b = -1$. In other words, the optimal tie-breaking rule for the designer is to resolve ties in favor of the weaker agent.

Now let's suppose the designer is constrained in that it should be ex-ante fair. In the case where agents are symmetric so that $V_1 = V_2 = V$, it is clear that the objective is increasing for $b > 0$ and decreasing for $b < 0$. And also, it is symmetric around $b = 0$. Thus, the optimal tie-breaking rule among those that are ex-ante fair would bias the contest in favor of one of the two agents with equal probability.

□

Lemma 9. *In a sudden death contest with symmetric agents $2 \geq V \geq \frac{3}{2}$, the expected effort from running a fair tie-breaker after the contest is higher than that from running it before the contest.*

Proof. In the sudden death contest game, we have $\frac{\partial p_i}{\partial x_i} = \frac{(1 - x_{-i})x_{-i}}{(x_i + x_{-i} - x_i x_{-i})^2}$, $\frac{\partial p_0}{\partial x_i} = \frac{x_{-i}^2}{(x_i + x_{-i} - x_i x_{-i})^2}$ and $c'(x) = 1$. Plugging these into the two first order conditions in equations 1 and 2 we get that the NE satisfies:

$$V_1\left(\frac{x_2(1-x_2+qx_2)}{(x_1+x_2-x_1x_2)^2}\right)=1$$

$$V_2\left(\frac{x_1(1-qx_1)}{(x_1+x_2-x_1x_2)^2}\right)=1$$

Consider the case where $V_1 = V_2$ and $q = \frac{1}{2}$. In this case, the unique symmetric equilibrium takes the form

$$x^* = 1-\sqrt{1-\frac{V}{2}}$$

Now suppose $q = 0$. In this case, the equilibrium satisfies $x_1 = x_2(1-x_2)$ and so the total effort in a pure strategy NE with $q = 0$ is bounded above by $\max_{x_2} x_2 + x_2(1-x_2) = \max_{x_2} x_2(2-x_2) = 1$. Thus, the effort in a Nash equilibrium under a biased contest ($q = 0$) must be less than that under a fair contest ($q = \frac{1}{2}$) if

$$2-2\sqrt{1-\frac{V}{2}} \geq 1 \iff V \geq \frac{3}{2}$$

Thus, we know that if the valuation is quite high, the principal would certainly prefer running a fair tie-breaker after the contest than biasing it by running a tie-breaker before the contest. □