TTC Domains

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Abstract

We study the object reallocation problem under strict preferences. On the unrestricted domain, Ekici (2024) showed that the Top Trading Cycles (TTC) mechanism is the unique mechanism that is individually rational, pair efficient, and strategyproof. We introduce a richness property on preference domains—the *top-two condition*—and show that this characterization extends to all domains satisfying it. The condition requires that within any subset of objects, if two objects can each be most-preferred, they can also be ranked as the top two (in either order). We further show that almost all domains failing the top-two condition for a triple or quadruple of objects admit non-TTC mechanisms satisfying the axioms. These results unify prior findings on specific domains, demonstrate the robustness of Ekici (2024)'s characterization, and suggest a minimal richness requirement that may underlie it.

1 Introduction

This paper studies the object reallocation problem, introduced by Shapley and Scarf (1974), from a mechanism design perspective. A group of agents, each endowed with an indivisible object, has strict, privately known preferences over all objects. A mechanism determines how the objects are reallocated based on the agents' reported preferences.

The Top Trading Cycles (TTC) mechanism is fundamental to the object reallocation problem. Shapley and Scarf (1974) proposed TTC (credited to David Gale) as an algorithm for finding an allocation in the core, and Roth and Postlewaite (1977) later proved that it produces the unique core allocation. Focusing on incentives, Roth (1982) showed that the TTC mechanism is strategyproof. In a seminal result, Ma (1994) characterized TTC as the unique mechanism satisfying individual rationality, Pareto efficiency, and strategyproofness. Alternative proofs were later provided by Svensson (1999), Anno (2015), Sethuraman (2016), and Bade (2019). More recently, Ekici (2024) strengthened this characterization by replacing Pareto efficiency with pair efficiency, a substantially weaker requirement that only rules out mutually beneficial swaps between two agents. Ekici and Sethuraman (2024) offer a short proof of this result. A growing literature examines whether these characterizations extend to restricted domains, such as single-peaked or single-dipped preferences

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(Bade (2019), Tamura (2023), Hu and Zhang (2024)).[1]

We introduce a richness property on preference domains, the *top-two condition*, and show that Ekici (2024)'s characterization extends to all domains satisfying it. The condition requires that within any subset of objects, if two objects can each be most-preferred, they can also be ranked as the top two (in either order). This property ensures that agents in a trading cycle can report their endowment as their second-most preferred remaining object—a key step in all existing proofs of Ekici (2024)'s and even Ma (1994)'s characterizations. At such profiles, while Pareto efficiency directly implies that trading cycles must be executed, Ekici (2024) and Ekici and Sethuraman (2024) exploit the richness of the unrestricted domain to establish the same result under pair efficiency. In comparison, we show that this conclusion follows with no additional richness beyond the top-two condition, thereby identifying a broad class of domains—henceforth, *TTC domains*—where Ekici (2024)'s characterization holds.

The sufficiency of the top-two condition lets us recover and unify existing characterization results, as well as classify some important but previously unexplored domains as TTC domains. For example, the single-dipped domain (Tamura (2023), Hu and Zhang (2024)) and the single-peaked domain with two adjacent peaks (Tamura (2022)) satisfy the top-two condition and are therefore TTC domains. We also introduce the *partial agreement domain*, defined by preferences consistent with a predefined partial order.[2] This is relevant in settings where some objects dominate others and preferences violating this dominance are inconceivable. For any partial order, the induced partial agreement domain satisfies the top-two condition and is therefore a TTC domain.

We further show that essentially all domains failing the top-two condition for a subset of size three or four admit non-TTC mechanisms satisfying individual rationality, Pareto efficiency, and strategyproofness. As a consequence, the top-two condition is necessary for the three-object and four-object cases, and moreover, pair efficiency and Pareto efficiency are equivalent under individual rationality and strategyproofness in these cases. To our knowledge, no domain failing the top-two condition is known to be a TTC domain, and non-TTC mechanisms have been identified on some such domains, such as the single-peaked domain (Bade (2019), Tamura and Hosseini (2022)). Our results also imply that the circular domain (Kim and Roush (1980)) is a non-TTC domain. Taken together, these findings lead us to conjecture that the top-two condition is necessary in general, thereby yielding a full characterization of TTC domains. At the very least, it offers a simple and useful criterion for analyzing the object reallocation problem under restricted domains.[3]

[1]The TTC mechanism has also been characterized using axioms such as group strategyproofness (Bird (1984), Takamiya (2001)), independence of irrelevant rankings (Morrill (2013)), non-bossiness (Miyagawa (2002), Ehlers (2014)), and endowments-swapping-proofness (Fujinaka and Wakayama (2018)). Some related strands of literature focus on reallocation problems tailored to specific environments (Abdulkadiroğlu and Sönmez (1999), Roth, Sönmez, and Ünver (2004), Schummer and Vohra (2013)) and object allocation problems (Carroll (2014), Hylland and Zeckhauser (1979), Pápai (2000), Pycia and Ünver (2017)). Morrill and Roth (2024) survey this literature, highlighting the relevance of TTC in these environments.

[2]Nicolo and Rodriguez-Alvarez (2017), Fujinaka and Wakayama (2024) study a related domain, which they refer to as the common ranking domain.

[3]In a similar spirit, Alcalde and Barbera (1994) propose the top-dominance criterion for two-sided matching, and Aswal, Chatterji, and Sen (2003) introduce the notion of linked domains for social choice. A recent survey by Elkind, Lackner, and Peters (2022) reviews a wide range of domain restrictions, some of which may be relevant in our context.

2 Model

Preliminaries

Let $N = \{1, \dots, n\}$ be a finite set of agents. Let $O = \{o_1, \dots, o_n\}$ be a finite set of indivisible objects such that o_i denotes agent i's endowment. Agents have strict preferences over objects. We denote by $\mathcal{P}$ the set of all strict linear orders over O, and we let $\mathcal{D} \subset \mathcal{P}$ denote the preference domain. Let $P = (P_i)_{i \in N} \in \mathcal{D}^N$ denote a preference profile where $P_i \in \mathcal{D}$ denotes agent i's preference over O. Following standard convention, for $S \subset N$, we let $P_S = (P_i)_{i \in S}$, $P_{-S} = (P_i)_{i \in N \setminus S}$. For each $P_0 \in \mathcal{D}$, we denote by R_0 the "at least as desirable as" relation associated with P_0, i.e., for each pair $o, o' \in O$, $o \; R_0 \; o'$ if and only if either $o \; P_0 \; o'$ or $o = o'$. We refer to a set of agents, their endowments, and their preferences over these objects, as an *economy*.

An *allocation* $x : N \to O$ is a bijection that assigns to each agent an object. Let $\mathcal{X}$ be the set of allocations. For each $x \in \mathcal{X}$ and each $i \in N$, we denote by $x_i \in O$ the assignment of agent i under the allocation x. A *mechanism* $\varphi : \mathcal{D}^N \to \mathcal{X}$ associates with each preference profile $P \in \mathcal{D}^N$ an allocation $x \in \mathcal{X}$.

Axioms

We now introduce some standard properties of allocations and mechanisms.

Given $P \in \mathcal{D}^N$, an allocation $x \in \mathcal{X}$ is individually rational at P if for each $i \in N$, $x_i \; R_i \; o_i$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *individually rational* if for each $P \in \mathcal{D}^N$, $\varphi(P)$ is individually rational at P.

Given $P \in \mathcal{D}^N$, an allocation $x \in \mathcal{X}$ is Pareto efficient at P if there is no other allocation $y \in \mathcal{X}$ such that for each $i \in N$, $y_i \; R_i \; x_i$ and for some $j \in N$, $y_j \; P_j \; x_j$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *Pareto efficient* if for each $P \in \mathcal{D}^N$, $\varphi(P)$ is Pareto efficient at P. Given $P \in \mathcal{D}^N$, an allocation $x \in \mathcal{X}$ is pair efficient at P if there is no $i, j \in N$ such that $x_j P_i x_i$ and $x_i P_j x_j$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *pair efficient* if for each $P \in \mathcal{D}^N$, $\varphi(P)$ is pair efficient at P.

A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *strategyproof* if for any $P \in \mathcal{D}^N$, there is no $i \in N$ and $P_i' \in \mathcal{D}$ such that $\varphi_i(P_i', P_{-i}) \; P_i \; \varphi_i(P)$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *group strategyproof* if for any $P \in \mathcal{D}^N$, there is no $S \subset N$ and $P_S' \in \mathcal{D}^S$ such that for each $i \in S$, $\varphi_i(P_S', P_{-S}) \; R_i \; \varphi_i(P)$ and for some $j \in S$, $\varphi_j(P_S', P_{-S}) \; P_j \; \varphi_j(P)$.

Top Trading Cycles

We now describe the TTC mechanism. For any strict profile $P \in \mathcal{P}^N$, the TTC algorithm finds an allocation as follows:

1. Each agent points to the agent who owns their most-preferred object.
2. In the ensuing directed graph between the agents, there is at least one cycle. All agents in a cycle are assigned their most-preferred objects and leave the economy.
3. The algorithm repeats with the remaining agents and their endowments.

We let $TTC(P) \in \mathcal{X}$ denote the allocation that results from running this algorithm at profile $P \in \mathcal{P}^N$. For any domain $\mathcal{D} \subset \mathcal{P}$, we define the *TTC mechanism* $\varphi : \mathcal{D}^N \to \mathcal{X}$ as the mechanism that selects for any preference profile $P \in \mathcal{D}^N$ the allocation $\varphi(P) = TTC(P)$. From previous results on the unrestricted domain, TTC mechanism satisfies all the properties defined above.

Fact 1. *For any domain $\mathcal{D}$, the TTC mechanism is:*

1. *individually rational,*

2. *Pareto efficient (and hence, pair efficient),*

3. *group strategyproof (and hence, strategyproof).*

For the unrestricted domain $\mathcal{D} = \mathcal{P}$, there is, in fact, no other mechanism that satisfies even the weak combination of individual rationality, pair efficiency, and strategyproofness (Ekici (2024)). In general, we say a domain $\mathcal{D} \subset \mathcal{P}$ is a *TTC domain* if there is no mechanism, other than TTC, that satisfies individual rationality, pair efficiency, and strategyproofness on $\mathcal{D}$. Our goal is to provide a general characterization of such domains.

Notation

We will sometimes describe a preference $P_0 \in \mathcal{D}$ by listing the objects in O in the order specified by P_0. For example, $P_0 \in \mathcal{D}$ such that $o_k \; P_0 \; o_{k+1}$ for all k can be succinctly represented as $o_1 o_2 \ldots o_n$.

For any $P_0 \in \mathcal{D}$ and $O' \subset O$, we let $r_k\,(P_0, O')$ denote the object ranked k-th under P_0 restricted to O'. For example, $r_1(P_0, O')$ is the most-preferred object among O' according to P_0.

Similarly, for any $\mathcal{D}' \subset \mathcal{D}$ and $O' \subset O$, we let $r_k(\mathcal{D}', O')$ denote the set of objects that can be ranked k-th according to some $P_0 \in \mathcal{D}'$ restricted to O'. Formally,

$$r_k(\mathcal{D}', O') = \{o' \in O' : \text{there exists } P_0 \in \mathcal{D}' \text{ such that } r_k(P_0, O') = o'\}.$$

For example, $r_1(\mathcal{D}, O')$ is the set of objects that can be most-preferred among O' according to preferences in $\mathcal{D}$.

3 Results

In this section, we present our results. We first introduce our richness condition, and then show how it offers a useful criterion for classifying domains as TTC domains or not TTC domains.

3.1 Top-two condition

Our key richness condition on preference domains requires that within any subset of objects, if two objects can each be most-preferred, they can also be ranked top two (in both possible orders).

Definition 1. A domain $\mathcal{D} \subset \mathcal{P}$ satisfies the *top-two condition* if for any $O' \subset O$ and any distinct $a, b \in r_1(\mathcal{D}, O')$, there exists a $P_0 \in \mathcal{D}$ such that

1. $a = r_1(P_0, O')$,

2. $b = r_2(P_0, O')$.

In other words, if a and b can each be most-preferred within the objects in O', there must be a preference where a is most-preferred and b is second most-preferred, and also a preference where b is most-preferred and a is second most-preferred.

We present some examples to illustrate the top-two condition. The unrestricted domain $\mathcal{D} = \mathcal{P}$ clearly satisfies the condition. Also, if $n \leq 2$, any $\mathcal{D} \subset \mathcal{P}$ satisfies the condition. Let $n = 3$ and consider the domains $\mathcal{D}_1 = \{o_1o_2o_3, o_2o_3o_1, o_2o_1o_3\}$ and $\mathcal{D}_2 = \{o_1o_2o_3, o_2o_3o_1, o_1o_3o_2\}$. In both cases, only o_1 and o_2 can be most-preferred among O. In $\mathcal{D}_1$, they can also be ranked top two (in both orders), but in $\mathcal{D}_2$, there is no preference where o_2 is most-preferred while o_1 is second-most preferred. Thus, $\mathcal{D}_1$ satisfies the top-two condition, while $\mathcal{D}_2$ does not. For another example, consider $n = 4$ and $\mathcal{D}_3 = \{o_1o_2o_3o_4, o_1o_3o_2o_4, o_2o_1o_4o_3, o_2o_4o_3o_1\}$. In this case, within $O' = \{o_1, o_3, o_4\}$, the objects o_1 and o_4 can be most-preferred, but there is no preference such that o_4 is most-preferred, and o_1 is second most-preferred. Thus, $\mathcal{D}_3$ does not satisfy the top-two condition.

Next, we discuss some important domains that have been studied in various contexts:

1. Suppose $n \geq 3$ and $\mathcal{D}$ is a *single-peaked domain* (Bade (2019)): This domain contains preferences that are single-peaked with respect to an underlying ordering of objects. WLOG, say $\mathcal{D} = \mathcal{D}^{SP}$, where $\mathcal{D}^{SP}$ is single-peaked with respect to the ordering $o_1 \to \cdots \to o_n$ so that

$$\mathcal{D}^{SP} = \{P_0 \in \mathcal{P} : o_p = r_1(P_0, O) \implies o_{k+1} \; P_0 \; o_k \text{ for } k < p \text{ and } o_k \; P_0 \; o_{k+1} \text{ for } k \geq p\}.$$

   Within any triple, the two extreme objects can be most-preferred, but they cannot be ranked top two. Thus, $\mathcal{D}$ does not satisfy the top-two condition.

2. Suppose $n \geq 3$ and $\mathcal{D}$ is a *single-peaked domain with two adjacent peaks* (Tamura (2022)): This domain further restricts the single-peaked domain so that only two adjacent objects can be most-preferred. WLOG, say $\mathcal{D} = \mathcal{D}^{SP-2}(p)$ for some $p \in \{1, \ldots, n-1\}$, where

$$\mathcal{D}^{SP-2}(p) = \{P_0 \in \mathcal{D}^{SP} : r_1(P_0, O) \in \{o_p, o_{p+1}\}\}.$$

   For any $O' \subset O$, only two adjacent objects can be most-preferred, and they can also be ranked as the top two. Thus, $\mathcal{D}$ satisfies the top-two condition.

3. Suppose $n \geq 3$ and $\mathcal{D}$ is a *single-dipped domain* (Tamura (2023)): This domain contains preferences that are single-dipped with respect to an underlying ordering of objects. WLOG, say $\mathcal{D} = \mathcal{D}^{SD}$, where $\mathcal{D}^{SD}$ is single-dipped with respect to the ordering $o_1 \to \cdots \to o_n$ so that

$$\mathcal{D}^{SD} = \{P_0 \in \mathcal{P} : o_d = r_n(P_0, O) \implies o_k \; P_0 \; o_{k+1} \text{ for } k < d \text{ and } o_{k+1} \; P_0 \; o_k \text{ for } k \geq d\}.$$

   For any $O' \subset O$, only the two objects at the extreme can be most-preferred, and they can also be ranked as the top two. Thus, $\mathcal{D}$ satisfies the top-two condition.

4. Suppose $n \geq 4$ and $\mathcal{D}$ is a *circular domain* (Kim and Roush (1980), Sato (2010)): This domain contains preferences described by the choice of a most-preferred object, and a clockwise or counterclockwise traversal along a fixed cyclic order on the set of objects. WLOG, say $\mathcal{D} = \mathcal{D}^C$ where $\mathcal{D}^C$ is circular with respect to the cyclic order $o_1 \to o_2 \to \cdots \to o_n \to o_1$ so that

$$D^C = \{P_0 \in \mathcal{P} : o_p = r_1(P_0, O) \implies P_0 \in \{o_p \ldots o_n o_1 \ldots o_{p-1}, o_p \ldots o_1 o_n \ldots o_{p+1}\}\}.$$

Within any quadruple, two non-adjacent objects can be most-preferred, but they cannot be ranked as the top two. Thus, $\mathcal{D}$ does not satisfy the top-two condition.

5. Suppose n is arbitrary and $\mathcal{D}$ is a *partial agreement domain*: This domain contains preferences that are consistent with some fixed partial dominance relation over the objects. Formally, say $\mathcal{D} = \mathcal{D}^{PA}(\succ)$ for some partial order $\succ$ on O, where

$$\mathcal{D}^{PA}(\succ) = \{P_0 \in \mathcal{P} : \text{ for all } a, b \in O, a \succ b \implies a \; P_0 \; b\}.$$

For any $O' \subset O$, only the undominated objects can be most-preferred, and any such pair can also be ranked as top two. Thus, this domain satisfies the top-two condition.

3.2 Top-two is sufficient

Our first result establishes that the top-two condition is sufficient for a preference domain to qualify as a TTC domain.

Theorem 1. *If $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition, then TTC is the unique mechanism that is individually rational, pair efficient, and strategyproof on $\mathcal{D}$.*

To prove this result, we show that at any preference profile, the top trading cycles must be executed. The idea is that, at any stage, agents involved in a trading cycle can report their endowment as their second-most preferred object among the remaining objects. At such a profile, individual rationality implies that either all these agents be assigned their endowment or the cycle be executed. While Pareto efficiency immediately rules out the former, the core of our argument lies in showing that the same conclusion follows even under the weaker requirement of pair efficiency. Specifically, we show in Lemma 1 that if these agents are instead assigned their endowments, then an agent in the cycle can obtain any object in the cycle (except their most-preferred object) by reporting it as their second-most preferred object, ultimately violating pair efficiency. From this point, we use individual rationality and strategyproofness repeatedly to conclude that the trading cycle must also be executed at the original profile. We now present the full proof of Theorem 1.

Proof. Suppose $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition, and $\varphi : \mathcal{D}^N \to \mathcal{X}$ is a mechanism that is individually rational, pair efficient, and strategyproof on $\mathcal{D}$. Consider any arbitrary preference profile $P \in \mathcal{D}^N$ and let $x = TTC(P)$. We will show that $\varphi(P) = x$.

Suppose $S \subset N$ denotes a subset of agents who would form a cycle and trade endowments in the first round of TTC at P. Notice that for any $i \in S$, it must be that both $x_i, o_i \in r_1(\mathcal{D}, O)$, and since $\mathcal{D}$ satisfies the top-two condition, we can find a $P'_i \in \mathcal{D}$ such that

1. $x_i = r_1(P'_i, O)$,
2. $o_i = r_2(P'_i, O)$.

We now focus on the profile (P'_S, P_{-S}). By individual rationality, it must be that either

1. $\varphi_i(P'_S, P_{-S}) = x_i$ for all $i \in S$ or
2. $\varphi_i(P'_S, P_{-S}) = o_i$ for all $i \in S$.

From here, the key is to show that the second case cannot hold, so that $\varphi_i(P'_S, P_{-S}) = x_i$ for all $i \in S$. If φ is Pareto efficient, this follows immediately from definition. We show that even if φ is pair efficient, under individual rationality and strategyproofness, this must be the case.

Lemma 1. *$\varphi_i(P'_S, P_{-S}) = x_i$ for all $i \in S$.*

Proof. Label the agents $S = \{i_1, i_2, \ldots, i_{|S|}\}$ so that i_s most prefers the endowment of i_{s+1} under P'_{i_s}, and $i_{|S|}$ most prefers the endowment of i_1:

P'_{i_1}	P'_{i_2}	P'_{i_3}	.	$P'_{i_{\lvert S\rvert-1}}$	$P'_{i_{\lvert S\rvert}}$
o_{i_2}	o_{i_3}	o_{i_4}	.	$o_{i_{\lvert S\rvert}}$	o_{i_1}
o_{i_1}	o_{i_2}	o_{i_3}	.	$o_{i_{\lvert S\rvert-1}}$	$o_{i_{\lvert S\rvert}}$
.	.	.	.	.	.
.	.	.	.	.	.

Suppose towards a contradiction that $\varphi_i(P'_S, P_{-S}) = o_i$ for all $i \in S$. We construct a sequence of profiles, which differ in only i_1's preference. For each $k \in \{3, 4, \ldots, |S|\}$, let $P^k_{i_1} \in \mathcal{D}$ be such that

$$r_1(P^k_{i_1}, O) = o_{i_2} \text{ and } r_2(P^k_{i_1}, O) = o_{i_k},$$

and define the preference profile

$$P^k = (P^k_{i_1}, P'_{S\setminus\{i_1\}}, P_{-S}).$$

These preferences are illustrated here:

$P^k_{i_1}$	P'_{i_2}	P'_{i_3}	.	$P'_{i_{\lvert S\rvert-1}}$	$P'_{i_{\lvert S\rvert}}$
o_{i_2}	o_{i_3}	o_{i_4}	.	$o_{i_{\lvert S\rvert}}$	o_{i_1}
o_{i_k}	o_{i_2}	o_{i_3}	.	$o_{i_{\lvert S\rvert-1}}$	$o_{i_{\lvert S\rvert}}$
.	.	.	.	.	.
.	.	.	.	.	.

We now show that for any $k \in \{3, 4, \ldots, |S|\}$,

$$\varphi_{i_1}(P^k) = o_{i_k}.$$

To begin, consider $k = |S|$. Suppose $\varphi_{i_1}(P^{|S|}) \notin \{o_{i_2}, o_{i_{|S|}}\}$. Then i_1 can misreport its preference to be such that it most prefers $o_{i_{|S|}}$, followed by o_{i_1}. By individual rationality and pair efficiency, i_1 must get $o_{i_{|S|}}$ with this misreport, and strategyproofness would be violated. Thus, $\varphi_{i_1}(P^{|S|}) \in \{o_{i_2}, o_{i_{|S|}}\}$. Further, by strategyproofness, $\varphi_{i_1}(P^{|S|}) \neq o_{i_2}$, and thus, $\varphi_{i_1}(P^{|S|}) = o_{i_{|S|}}$.

Now consider any $k \in \{3, \ldots, |S| - 1\}$ and assume $\varphi_{i_1}(P^{k+1}) = o_{i_{k+1}}$. Suppose $\varphi_{i_1}(P^k) \notin \{o_{i_2}, o_{i_k}\}$. Then i_1 can misreport its preference to be such that it most prefers o_{i_k}, followed by $o_{i_{k+1}}$. Since i_1 can ensure $o_{i_{k+1}}$ (by reporting $P^{k+1}_{i_1}$), i_1's assignment with this misreport should be in $\{o_{i_k}, o_{i_{k+1}}\}$. But if i_1 gets $o_{i_{k+1}}$, by individual rationality, i_k should get o_{i_k}, which violates pair efficiency. Therefore, i_1 must get o_{i_k} with this misreport, and strategyproofness would be violated. Thus, $\varphi_{i_1}(P^k) \in \{o_{i_2}, o_{i_k}\}$. Further, by strategyproofness, $\varphi_{i_1}(P^k) \neq o_{i_2}$, and thus, $\varphi_{i_1}(P^k) = o_{i_k}$.

Thus, for any $k \in \{3, 4, \ldots, |S|\}$, $\varphi_{i_1}(P^k) = o_{i_k}$. In particular, $\varphi_{i_1}(P^3) = o_{i_3}$. By individual rationality, $\varphi_{i_2}(P^3) = o_{i_2}$. But then φ violates pair efficiency, which is a contradiction. □

Going back to the proof of the Theorem, we now have that $\varphi_i(P'_S, P_{-S}) = x_i$ for all $i \in S$.

Fix any $j \in S$. By strategyproofness, $\varphi_j(P_j, P'_{S\setminus\{j\}}, P_{-S}) = x_j$, and by individual rationality, for each $i \in S$,

$$\varphi_i(P_j, P'_{S\setminus\{j\}}, P_{-S}) = x_i.$$

Now suppose for any $T \subset S$ where $|T| \le k < |S|$, we have that for each $i \in S$,

$$\varphi_i(P_T, P'_{S\setminus T}, P_{-S}) = x_i.$$

Fix any T of size $k + 1$ and any $j \in T$. By strategyproofness, $\varphi_j(P_T, P'_{S\setminus T}, P_{-S}) = x_j$, and in fact, for any $i \in T$, $\varphi_i(P_T, P'_{S\setminus T}, P_{-S}) = x_i$. Now there must be some $i \in T$ such that $x_i = o_r$ where $r \in S \setminus T$. By individual rationality, it must be that for each $i \in S$, $\varphi_i(P_T, P'_{S\setminus T}, P_{-S}) = x_i$.

It follows by induction that for each $i \in S$,

$$\varphi_i(P_S, P_{-S}) = \varphi_i(P) = x_i.$$

Now we can iteratively apply this argument to agents in the second cycle, third cycle, and so on, to get that for each $i \in N$, $\varphi_i(P) = x_i$.

It follows that $\varphi(P) = TTC(P)$ for all $P \in \mathcal{D}^N$, and thus, φ must be the TTC mechanism. □

Our proof of Theorem 1 provides an alternative to existing proofs of Ekici (2024)'s characterization (Ekici (2024), Ekici and Sethuraman (2024)), relying only on minimal richness of the unrestricted domain. In our context, the proof in Ekici (2024) requires the domain to satisfy a *top-three condition* (any three objects that can each be most-preferred can also be the top-three most-preferred objects in all possible orders). The shorter proof by Ekici and Sethuraman (2024) relies on an even stronger *top-k condition*, which requires this property for every k. Notably, existing proofs of Ma (1994)'s characterization with Pareto efficiency also require the domain to satisfy at least the top-two condition. In comparison, our proof establishes a stronger characterization with pair efficiency assuming only the top-two condition.

Theorem 1 enables the classification of several important restricted domains as TTC domains. In particular, the single-peaked domain with two adjacent peaks and the single-dipped domain satisfy the top-two condition, and are therefore TTC domains. The partial agreement domain, previously unexplored, satisfies the top-two condition, leading to the following corollary.

Corollary 1. *Suppose $\mathcal{D} = \mathcal{D}^{PA}(\succ)$ is a partial agreement domain. There is a unique individually rational, pair efficient, and strategyproof mechanism on $\mathcal{D}$.*

More generally, even though Theorem 1 assumes a common domain $\mathcal{D}$, our proof extends to settings with heterogeneous domain restrictions $(\mathcal{D}_i)_{i\in N}$.[4] While we do not provide a formal sufficient condition, if the domains are jointly rich enough to construct the profiles in Lemma 1, our argument applies, and TTC remains the only mechanism that is individually rational, pair efficient, and strategyproof.

[4] Note that each $\mathcal{D}_i$ individually satisfying the top-two condition is not sufficient for the characterization. For example, consider $n = 3$ with $\mathcal{D}_1 = \{o_2o_1o_3\}, \mathcal{D}_2 = \{o_3o_2o_1\}$, and $\mathcal{D}_3 = \{o_1o_3o_2\}$. In this case, only one preference profile exists, and the mechanism that returns the endowment allocation satisfies all three axioms.

3.3 Top-two is necessary?

We now turn to the question of whether the top-two condition is necessary for a domain to qualify as a TTC domain. Equivalently: if a domain fails the top-two condition, does there exist a non-TTC mechanism satisfying the three axioms? While we cannot establish existence for all such domains, we construct such a mechanism for essentially every domain that fails the top-two condition for a triple or quadruple of objects.

Definition 2. A domain $\mathcal{D} \subset \mathcal{P}$ *fails the top-two condition for* O' if there exist $a, b \in r_1(\mathcal{D}, O')$ such that for all $P_0 \in \mathcal{D}$,

$$a = r_1(P, O') \implies b \neq r_2(P_0, O').$$

If $\mathcal{D}$ fails the top-two condition, then there exists some $O' \subset O$ for which $\mathcal{D}$ fails the top-two condition.

When the failure occurs for the full set O, the following result gives a non-TTC mechanism that satisfies the axioms for $n \leq 4$.

Proposition 1. *Suppose $n \leq 4$ and $\mathcal{D} \subset \mathcal{P}$ fails the top-two condition for $O' = O$. Then, there exists a non-TTC mechanism that is individually rational, Pareto efficient, and strategyproof on $\mathcal{D}$.*

Proof. WLOG, assume $\mathcal{D}$ is such that $o_1, o_2 \in r_1(\mathcal{D}, O)$, and for any $P_0 \in \mathcal{D}$,

$$o_2 = r_1(P_0, O) \implies o_1 \neq r_2(P_0, O).$$

Further assume there exists $P_0 \in \mathcal{D}$ such that $o_2 \; P_0 \; o_3 \; P_0 \cdots P_0 \; o_n$ (relabeling if needed).

Define a subset of preference profiles:

$$Diff = \{P \in \mathcal{D}^N : r_1(P_1, O) = o_2, \text{ and for } i \geq 2, \; r_1(P_i, \{o_{i-1}, o_i, \ldots, o_n\}) = o_{i-1}\}.$$

Define the mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ as:

$$\varphi(P) = \begin{cases} TTC(P) & \text{if } P \notin Diff, \\ (o_k, o_1, o_2, \ldots, o_{k-1}, TTC(P_{\{k+1,\ldots,n\}} | \{o_{k+1}, \ldots, o_n\})) & \text{if } P \in Diff \text{ and } o_k = r_2(P_1, O). \end{cases}$$

For $P \in Diff$, $\varphi(P) \neq TTC(P)$. It is straightforward to verify that φ is individually rational and Pareto efficient. We will now show that φ is strategyproof for $n \leq 4$.

Let $P \in \mathcal{D}^N$ be any profile. Consider any agent $i \in N$ and a potential misreport $P'_i \in \mathcal{D}$. If both $P, (P'_i, P_{-i}) \notin Diff$, then by the strategyproofness of TTC, $\varphi_i(P) \; R_i \; \varphi_i(P'_i, P_{-i})$. Similarly, if both $P, (P'_i, P_{-i}) \in Diff$, we again have $\varphi_i(P) \; R_i \; \varphi_i(P'_i, P_{-i})$. Thus, the only misreports to consider are those that switch membership in $Diff$. In such cases, P_{-i} must be such that agent i's report determines membership in $Diff$. We analyze these cases next.

1. $i = 1$: By definition of $Diff$, P_{-1} is such that for any $P_1 \in \mathcal{D}$, $\varphi_1(P) = r_1(P_1, O \setminus \{o_2\})$. Thus, there is never an incentive for agent 1 to misreport.
2. $i = 2$: For all $P \in \mathcal{D}^N$, $\varphi_2(P) = TTC_2(P)$. Thus, there is never an incentive for agent 2 to misreport.

3. $i = 3$: By definition of $Diff$, P_{-3} is such that for any $P_3 \in \mathcal{D}$, $\varphi_3(P) = r_1(P_3, O \setminus \{o_1\})$. Thus, there is never an incentive for agent 3 to misreport.

4. $i = 4$: Fix $n = 4$.[5] By definition of $Diff$, P_{-4} is such that for any $P_4 \in \mathcal{D}$, $f_4(P) \in \{o_3, o_4\}$. We consider two subcases:

   (a) P_4 is such that $(P_4, P_{-4}) \notin Diff$: This means that $o_4\ P_4\ o_3$, and $\varphi_4(P) = o_4 = r_1(P_4, O \setminus \{o_1, o_2\})$.

   (b) P_4 is such that $(P_4, P_{-4}) \in Diff$: This means that $o_3\ P_4\ o_4$, and $\varphi_4(P) \in \{o_3, o_4\}$ depending upon $r_2(P_1, O)$.

   It follows that there is no incentive for agent 4 to misreport in either of the subcases.

Thus, for $n \leq 4$, φ is strategyproof. □

We now leverage Proposition 1 to handle domains that fail the top-two condition for triples or quadruples. As an illustration, recall $\mathcal{D}_3 = \{o_1o_2o_3o_4, o_1o_3o_2o_4, o_2o_1o_4o_3, o_2o_4o_3o_1\}$, which fails the top-two condition for $O' = \{o_1, o_3, o_4\}$ but not for O. We can define a mechanism that coincides with TTC except when agent 2 most prefers o_2, in which case we apply the three-agent mechanism from Proposition 1, after appropriate relabeling on agents $\{1, 3, 4\}$ and objects $\{o_1, o_3, o_4\}$. This yields a non-TTC mechanism satisfying all three axioms on $\mathcal{D}_3$. The same idea leads to the following general result.

Theorem 2. *Suppose $\mathcal{D} \subset \mathcal{P}$ fails the top-two condition for an $O' \subset O$ such that:*

1. $|O'| \leq 4$,
2. *For every $o \in O \setminus O'$, $o \in r_1(\mathcal{D}, O' \cup \{o\})$.*

Then there exists a non-TTC mechanism that is individually rational, Pareto efficient (hence, pair efficient), and strategyproof on $\mathcal{D}$.

Proof. WLOG, let $O' = \{o_1, o_2, \ldots, o_k\}$. For $i \in \{k+1, \ldots, n\}$, define

$$\mathcal{D}_i = \{P_0 \in \mathcal{D} : o_i = r_1(P_0, O' \cup \{o_i\})\},$$

and note that it is non-empty. Define $\varphi : \mathcal{D}^N \to \mathcal{X}$ by

$$\varphi(P) = \begin{cases} TTC(P), & \text{if } P_i \notin \mathcal{D}_i \text{ for some } i \in \{k+1, \ldots, n\}, \\ (\varphi'(P_{\{1,\ldots,k\}}|O'), TTC(P_{\{k+1,\ldots,n\}}|O \setminus O')), & \text{if } P_i \in \mathcal{D}_i \text{ for all } i \in \{k+1, \ldots, n\}, \end{cases}$$

where φ' is the mechanism defined in Proposition 1. It is straightforward to verify that φ is individually rational, Pareto efficient, and strategyproof on $\mathcal{D}$. □

[5] When $n > 4$, φ may not be strategyproof for $i = 4$. To see why, suppose $n = 5$, and $P \in \mathcal{D}^N$ is such that $r_1(P_3, \{o_3, o_4, o_5\}) = o_5, r_1(P_5, \{o_3, o_4, o_5\}) = o_3$ and $o_5\ P_4\ o_3\ P_4\ o_4$. Then $\varphi_4(P) = o_4$. By reporting $P_4' \in \mathcal{D}$ such that $o_3 = r_1(P_4', \{o_3, o_4, o_5\})$, we have $(P_4', P_{-4}) \in Diff$ and $\varphi_4(P_4', P_{-4}) = o_3$ when $r_2(P_1, O) = o_4$, yielding a better outcome.

Together, Proposition 1 and Theorem 2 imply that the top-two condition is necessary for $n \leq 4$. The $n = 3$ case follows directly from Proposition 1. For $n = 4$, let $O' \subset O$ be a subset of maximum cardinality for which the top-two condition fails. If $|O'| = 4$, Proposition 1 applies. If $|O'| = 3$, the maximality of O' ensures the second condition in Theorem 2 holds, so that the theorem applies. Thus, top-two is necessary for $n = 4$. Moreover, since our construction in Proposition 1 and Theorem 2 is Pareto efficient, we obtain the following:

Corollary 2. *Suppose $n \leq 4$ and $\mathcal{D} \subset \mathcal{P}$. The following are equivalent:*

1. *$\mathcal{D}$ satisfies the top-two condition.*
2. *There is a unique individually rational, Pareto efficient, and strategyproof mechanism on $\mathcal{D}$.*
3. *There is a unique individually rational, pair efficient, and strategyproof mechanism on $\mathcal{D}$.*

The scope of Theorem 2 and Corollary 2 is directly linked to Proposition 1: if Proposition 1 can be extended to $n \leq m$, then Theorem 2 holds with $|O'| \leq m$, and Corollary 2 extends to $n \leq m$.

Although Theorem 2 does not establish necessity in full generality, it is difficult to find domains failing the top-two condition that lie outside its scope; such domains are likely to be artificial and of limited practical relevance. In particular, it enables the classification of several important and structured domains as non-TTC domains. As noted earlier, the single-peaked domain fails the top-two condition for triples and is therefore not a TTC domain. The circular domain, previously unstudied in this context, fails the condition for quadruples, leading to the following corollary.

Corollary 3. *Suppose $n \geq 4$ and $\mathcal{D} = \mathcal{D}^C$ is the circular domain. There exists a non-TTC mechanism that is individually rational, Pareto efficient, and strategyproof mechanism on $\mathcal{D}$.*

4 Conclusion

We introduce a richness property on preference domains—the *top-two condition*—and establish it as a simple and powerful criterion for analyzing object reallocation problems under restricted domains. The condition requires that within any subset of objects, if two objects can each be most-preferred, they can also be ranked as the top two (in either order). We show that the characterization of TTC by Ekici (2024) (and hence by Ma (1994)) on the unrestricted domain extends to all domains satisfying this condition. We further show that these characterizations do not extend to domains failing the condition for small subsets, thereby establishing its necessity for such cases and suggesting its necessity more generally. Our findings provide a unifying perspective on previously studied domain restrictions, such as single-peaked and single-dipped domains, while also classifying previously unexplored domains, such as the partial agreement and circular domains, as either TTC or non-TTC domains. These results open several directions for future research. An immediate question is whether the top-two condition is necessary in full generality, and further work could explore analogous richness conditions for other characterizations of TTC based on axioms such as group strategyproofness.

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