

Optimal screening contests ^{*}

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Abstract

We study the optimal design of contests as screening devices. In an incomplete information environment, contest results reveal information about the quality of the participating agents at the cost of potentially wasteful effort put in by these agents. We are interested in finding contests that maximize the information revealed per unit of expected effort put in by the agents. In a model with linear costs of effort and privately known marginal costs, we find the Bayes-Nash equilibrium strategy for arbitrary prize structures ($1 = v_1 \geq v_2 \cdots \geq v_n = 0$) and show that the equilibrium strategy mapping marginal costs to effort is always a density function. It follows then that the expected effort under the uniform prior on marginal costs is independent of the prize structure. Restricting attention to a simple class of uniform prizes contests (top k agents get 1 and others get 0), we find that the optimal screening contest under the uniform prior awards half as many prizes as there are agents. For the power distribution $F(\theta) = \theta^p$ with $p \geq 1$, we conjecture that the number of prizes in the optimal screening contest is decreasing in p . In addition, we also show that a uniform prize structure is generally optimal for the standard objectives of maximizing expected effort of an arbitrary agent, most efficient agent and least efficient agent.

1 Introduction

There are many situations in which contests are used to screen participants. For instance, there exist organizations that conduct exams whose results participants can then use to signal their quality to employers or schools. One such example is Kaggle, an online community of data scientists that organizes competitions to solve data science challenges. The scientists who provide the best solutions are rewarded with digital medals and certificates which signal their quality to the market. There is also the popular signalling theory of education according to which students acquire education credentials to send a signal about their ability level to employers. The theory motivates the view of college as being a screening contest where students compete for better signals (grades or GPA) which they can then use to get better offers in the market. In such screening contests, it is natural to think that the designer cares

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about the information revealed by the contest results about the quality of the participating agents. In addition, the designer may also care about the effort exerted by the participating agents towards the objective of getting better signals which might be wasteful (as in the signalling theory of education) or productive (if it improves their ability). In this paper, we focus on settings where this effort is wasteful and consider the problem of designing a contest to maximize the information revealed per unit expected effort put in by the agents.

Formally, we consider a contest design problem for n agents where the designer commits to a prize structure $v = (v_1, v_2, \dots, v_n)$ such that $v_1 = 1, v_n = 0$, and $v_i \geq v_{i+1}$ for all $i \in \{1, 2, \dots, n-1\}$. The n agents have linear costs of effort and are awarded according to how much relative effort they put in. Agents differ in their marginal costs of effort which is a measure of their quality (or inefficiency) and is their private information. We assume that these marginal costs are drawn iid according to cdf F on $[0, 1]$. In this environment, we first find the symmetric Bayes-Nash equilibrium strategy mapping marginal costs to effort levels for arbitrary prize structures and priors. The equilibrium function has a couple of interesting properties. First, it is linear in each of the n prizes v_i . This property lends a simple structure to the optimal contests we derive later. Second, it turns out that for any prize structure and under a mild condition on the prior, the equilibrium effort function is always a density function. This means that for any prior, changing the prize structure only redistributes the effort among the agents. While we do not have a good intuition for why this is the case, the observation helps provide intuition for some of our optimality results.

Before discussing contests as screening devices, we consider some standard objectives for contest design from the contest theory literature. We find the optimal contest for maximizing expected effort of an arbitrary agent, of the most efficient agent, and of the least efficient agent. The linearity of the objective in prizes implies that the optimal contest generally follows a uniform prize structure. These are prize structures in which the designer just chooses the number of prizes k to award so that $v_i = 1$ for $i \leq k$ and 0 otherwise. For the uniform prior, it follows from the fact that the equilibrium function is a density function that the expected effort remains constant irrespective of the prize structure. For the case where $F(\theta) = \theta^p$ with $p > 1$, we find that awarding $n - 1$ uniform prizes is optimal for maximizing both expected effort and expected minimum effort. Intuitively, under these priors and objectives, the designer puts more weight on the equilibrium effort of the less efficient agents which is higher when the designer awards more prizes. In contrast, for maximizing expected maximum effort, the number of prizes is typically interior.

For the screening objective, we consider a contest designer who wants to choose a prize structure to maximize the variance of posterior means (measure of information revealed) per unit expected effort put in by the agents. For this objective, we restrict attention to the class of uniform prize structures. As motivation, consider a university that organizes pass-fail exams that students take to signal their quality to the market. The university would like to design the contest so that its results are informative about the quality of the students.

At the same time, the university understands that the effort put in by the students towards getting better signals is otherwise wasteful and therefore, would like to design the contest so that this wasteful effort is minimized. To account for both desiderata simultaneously, we consider the objective of maximizing variance of posterior means per unit expected effort put in by the agents. For this objective, we find that under the uniform prior with n agents, awarding $\frac{n}{2}$ uniform prizes is optimal. For the power distribution $F(\theta) = \theta^p$ with $p > 1$, we conjecture that the optimal screening contest awards a decreasing number of prizes as we increase p , and that eventually, awarding just a single prize is optimal.

The problem of finding the optimal prize structure has been extensively studied in the literature for various objectives and environments. We believe ours is the first to consider the optimal design of contests as screening devices. The objective of maximizing effort is more standard in the literature. A vast literature considers the allocation of a fixed budget across multiple prizes in complete or incomplete information environments (Glazer and Harsanyi [16], Barut and Kovenock [2], Krishna and Morgan [20], Moldovanu and Sela [26, 27], Liu and Lu [24], Chawla et al. [4], Ales et al. [1]). Under linear effort costs, Moldovanu and Sela [26] find that it is optimal to allocate the entire budget to a single first prize. In contrast, our results suggest that when prizes are costless, it is optimal to award a prize to everyone except the agent who puts in the least effort. Fang et al. [10] obtain similar results in a complete information all-pay contest setting as they show that increased competition discourages effort. Another paper closely related to ours is Liu and Lu [25] who consider the problem of finding optimal uniform prize structures for maximizing effort but under different distributional assumptions. Under their assumptions, they find that both expected effort and expected maximum effort are single peaked in the number of prizes. They further find that maximizing expected highest effort requires a smaller number of prizes as compared to maximization of expected effort. In comparison, we find distributions under which the expected effort is monotone increasing in the number of prizes so that awarding $n - 1$ prizes is optimal. Surveys of the theoretical literature in contest theory can be found in Corchón [8], Vojnovic [33], Konrad et al. [19], Segev [31], Sisak [32].

The paper proceeds as follows. In section 2, we present the model. Section 3 finds the Bayes-Nash equilibrium strategy function and also the optimal prize structure for maximizing effort. In section 4, we find the optimal screening contest. Section 5 concludes.

2 Model

There are n risk-neutral agents competing for n prizes given by $v = (v_1, v_2, \dots, v_n)$ such that $1 = v_1 \geq v_2 \geq \dots \geq v_n = 0$. Given the prize structure, all agents simultaneously exert efforts and the agent who puts in the k th highest effort is awarded prize v_k . The cost of effort e_i for agent i is given by $\theta_i e_i$ where the marginal cost θ_i is agent's private information and is drawn iid according to cdf $F(\cdot)$ on $[0, 1]$ with density $f(\cdot) > 0$. So if agent i exerts effort e_i and gets prize v_k , its payoff is $v_k - \theta_i e_i$.

In this setup, we are interested in finding the prize structure for maximizing the information revealed by the contest (measured by variance of posterior means) per unit of effort put in by the agents. For the screening problem, we'll restrict attention to a simple and natural class of uniform prize structures. These are prize structures in which the designer just chooses the number of prizes k so that the prize vector takes the form $v = (1, 1, \dots, 1, 0, \dots, 0)$. Before discussing the screening problem for uniform prize structures, we first find the Bayes-Nash equilibrium strategy of the agents for arbitrary prize structure $v = (1, v_2, \dots, v_{n-1}, 0)$.

3 Equilibrium strategy

In this section, we focus on finding and analyzing the symmetric Bayes-Nash equilibrium strategy function.

Theorem 1. *In a contest with n agents, prizes $v = (1, v_2, \dots, v_{n-1}, 0)$ and prior cdf F , the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g(\theta) = \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt$$

The proof proceeds by assuming that $n-1$ agents are playing $g(\theta)$ where g is a decreasing function. Then, we find a player's optimal effort level at type θ by taking the first order condition. Plugging in $g(\theta)$ for the optimal level of effort in the condition gives the condition for $g(\theta)$ to be the symmetric Bayes-Nash equilibrium:

$$-f(\theta) \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1-F(\theta))^{n-i-1} F(\theta)^{i-1}] = \theta g'(\theta)$$

Using the boundary condition $g(1) = 0$ pins down the form of the function. We then check that the second order condition is satisfied. The full proof is in the appendix.

We now state a surprising and useful property of the equilibrium function.

Lemma 1. *In a contest with n agents, prizes $v = (1, v_2, \dots, v_{n-1}, 0)$ and prior cdf F , if the equilibrium function is such that $\lim_{\theta \rightarrow 0} \theta g(\theta) = 0$, then it is a density function. That is,*

$$\int_0^1 g(\theta) d\theta = 1$$

The proof proceeds as follows. Under the assumption, we know that

$$\int_0^1 \theta g'(\theta) d\theta = \theta g(\theta) \Big|_0^1 - \int_0^1 g(\theta) d\theta = - \int_0^1 g(\theta) d\theta \quad (\text{Assuming } \lim_{\theta \rightarrow 0} \theta g(\theta) = 0)$$

Given the form of the equilibrium condition, we can find

$$\int_0^1 \theta g'(\theta) d\theta = \int_0^1 -f(\theta) \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1-F(\theta))^{n-i-1} F(\theta)^{i-1}] d\theta$$

and it turns out to be equal to $v_n - v_1 = -1$. It follows then that the equilibrium function is always a density function.

We now show that the condition $\lim_{\theta \rightarrow 0} \theta g(\theta) = 0$ is satisfied for the power distribution prior $F(\theta) = \theta^p$ with $p \geq 1$.

Lemma 2. *In a contest with n agents, prizes $v = (1, v_2, \dots, v_{n-1}, 0)$ and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$, the equilibrium function is such that $\lim_{\theta \rightarrow 0} \theta g(\theta) = 0$.*

Proof. We basically want to show that for any $i \in \{1, 2, \dots, n-1\}$,

$$\lim_{\theta \rightarrow 0} \theta \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt = \lim_{\theta \rightarrow 0} \theta \int_{\theta^p}^1 (1-t)^{n-i-1} t^{i-1-\frac{1}{p}} dt = 0$$

Observe that for $i \geq 2$, the function is bounded above by $\theta(1-\theta^p)$ which goes to 0 as $\theta \rightarrow 0$. So now let's look at $i = 1$. In this case, we have

$$\theta \int_{\theta^p}^1 (1-t)^{n-2} t^{-\frac{1}{p}} dt \leq \theta \int_{\theta^p}^1 \frac{(1-t)^{n-2}}{t} dt$$

and we can show that $\lim_{\theta \rightarrow 0} \theta \int_{\theta^p}^1 \frac{(1-t)^{n-2}}{t} dt = 0$ □

It follows from Lemma 2 that the result of Lemma 1 applies for the case where $F(\theta) = \theta^p$ with $p \geq 1$. In particular, we get that for the uniform prior, the expected effort is constant at 1 irrespective of the prize structure $(1, v_2, \dots, v_{n-1}, 0)$.

Corollary 1. *In a contest with n agents, prizes $v = (1, v_2, \dots, v_{n-1}, 0)$ and uniform prior cdf $F(\theta) = \theta$, the expected effort is constant at 1. That is,*

$$\int_0^1 g(\theta) f(\theta) d\theta = \int_0^1 g(\theta) d\theta = 1$$

A standard objective considered in the literature on optimal contest theory is finding prize structure that maximize expected effort. The next result shows that the optimal prize structure v under the power distribution prior awards uniform prizes of 1 to everyone except the worst agent.

Theorem 2. *In a contest with n agents and prior cdf $F(\theta) = \theta^p$ with $p > 1$, the expected effort is maximized by $n-1$ uniform prizes ($v^* = (1, 1, \dots, 1, 0)$).*

The proof proceeds by showing that in the general case, the expected effort equals

$$EE(n, v, F) = \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1-t)^{n-i-1} t^i}{F^{-1}(t)} dt$$

Importantly, the objective is linear in all the prizes. Thus, we only have to check if the marginal gain from a prize is positive or negative to determine how much we should set it at. Given the above expression, we can show that the following equivalence holds:

$$\frac{\partial EE(n, v, F)}{\partial v_i} \geq 0 \iff \frac{n-i}{i-1} \geq \frac{\int_0^1 \frac{(1-t)^{n-i} t^{i-1}}{F^{-1}(t)} dt}{\int_0^1 \frac{(1-t)^{n-i-1} t^i}{F^{-1}(t)} dt}$$

Now for $F(\theta) = \theta^p$ with $p > 1$, the condition on the right holds and this gives us the result that awarding $n-1$ uniform prizes is optimal.

Intuitively, we know from Lemma 1 that the equilibrium effort function is always a density function. And as we increase the number of prizes, the effort essentially shifts from low marginal cost agents to high marginal cost agents. That is, the more efficient agents put in lower levels of effort and the less efficient agents put in more levels of effort as we increase the number of prizes. This is illustrated in Figure 1. For convex cdf's of the form $F(\theta) = \theta^p$ with $p > 1$, we have more of high marginal cost agents and hence, we care more about the effort put in by these less efficient agents. As a result, the optimal prize structure awards the maximum number of prizes in order to incentivize the less efficient majority.

Another standard objective considered in the contest theory literature is to maximize the expected minimum effort. For this objective, our model again suggests that awarding $n-1$ uniform prizes is optimal.

Theorem 3. *In a contest with n agents and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$, the expected minimum effort is maximized by $n-1$ uniform prizes $v^* = (1, 1, \dots, 1, 0)$.*

The proof follows exactly the same steps as the proof of Theorem 2. We integrate the equilibrium strategy function with respect to density of the maximum marginal cost. The intuition provided for Theorem 2 also extends to this result and is in fact, even stronger here. This is because the objective itself now gives more weight to agents with high marginal costs and so we get that even under the uniform prior, awarding $n-1$ uniform prizes is strictly optimal for maximizing expected minimum effort.

Lastly, let's also consider the objective of maximizing the expected maximum effort.

Theorem 4. *In a contest with n agents and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$, the expected maximum effort is maximized at some uniform prize structure $v = (1, \dots, 1, 0, \dots, 0)$.*

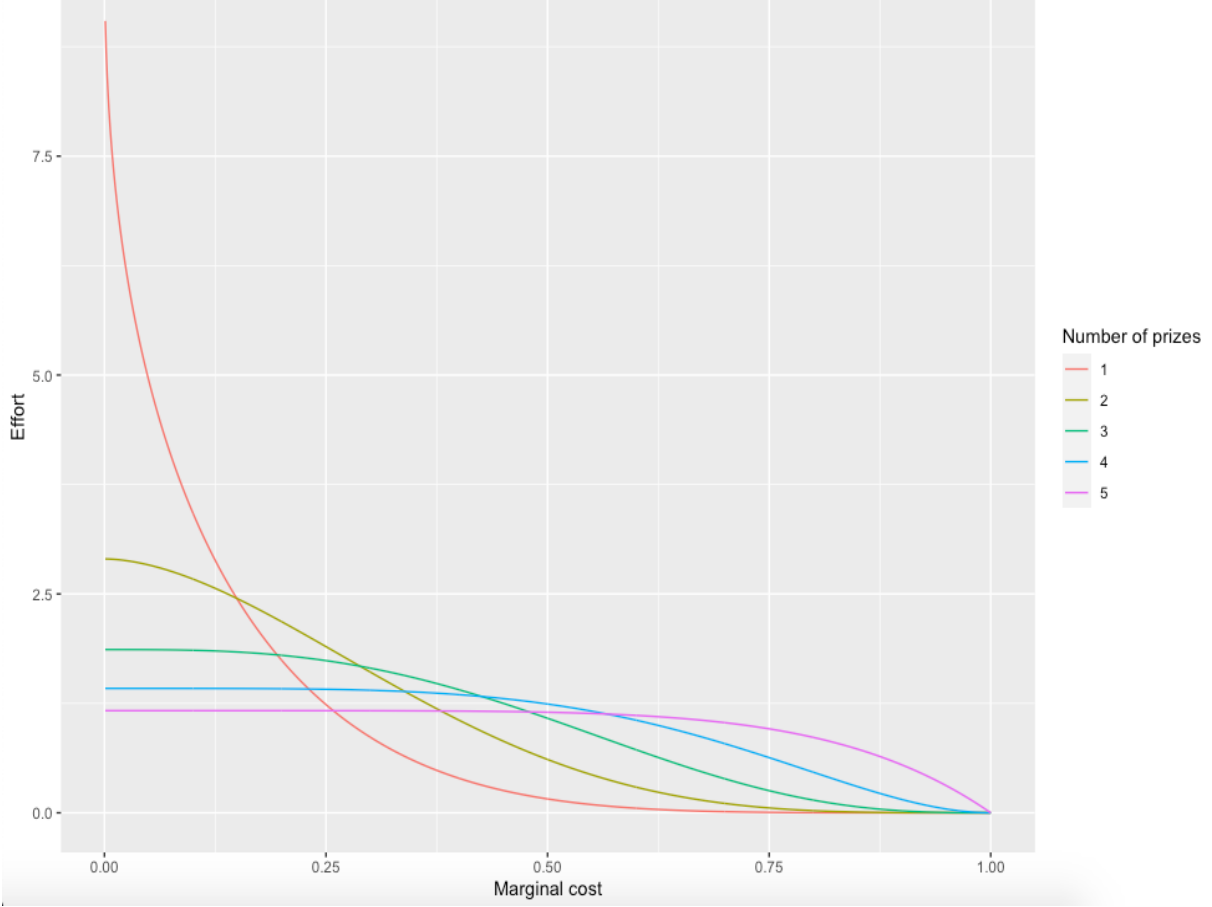


Figure 1: The equilibrium effort function for $n = 6$, $F(\theta) = \theta^{1.4}$ under different no. of prizes

The proof again follows the same steps as those for the proof of Theorem 2. We integrate the equilibrium strategy function with respect to density of the minimum marginal cost. This leads to the following condition on the marginal effect on expected maximum effort of prize v_i :

$$\frac{\partial EMAXE(n, v, F)}{\partial v_i} \geq 0 \iff \frac{i - 1 - \frac{1}{p}}{i - 1} \geq \frac{\left(1 - \prod_{j=1}^n \frac{n - i + j}{n - 1 - \frac{1}{p} + j}\right)}{\left(1 - \prod_{j=1}^n \frac{n - i + j - 1}{n - 1 - \frac{1}{p} + j}\right)}$$

Observe that now, the objective gives more weight to agents with low marginal costs and so we get that awarding an interior number of prizes is typically optimal. We believe that the optimal number of prizes increases as we increase p . This is again based on the same intuition that as we increase p , the proportion of high marginal cost agents increases and so even the agent who puts in the maximum effort is likely to have a high marginal cost. And to incentivize agents with high marginal costs to put in more effort, it is better to increase

prizes. Figure 2 supports this intuition. It plots the expected maximum effort as a function of number of prizes for the case of $n = 15$ agents under different values of p . We can see that the optimal number of prizes is interior and actually increases as we increase p . We believe this is true but we do not have a proof yet.

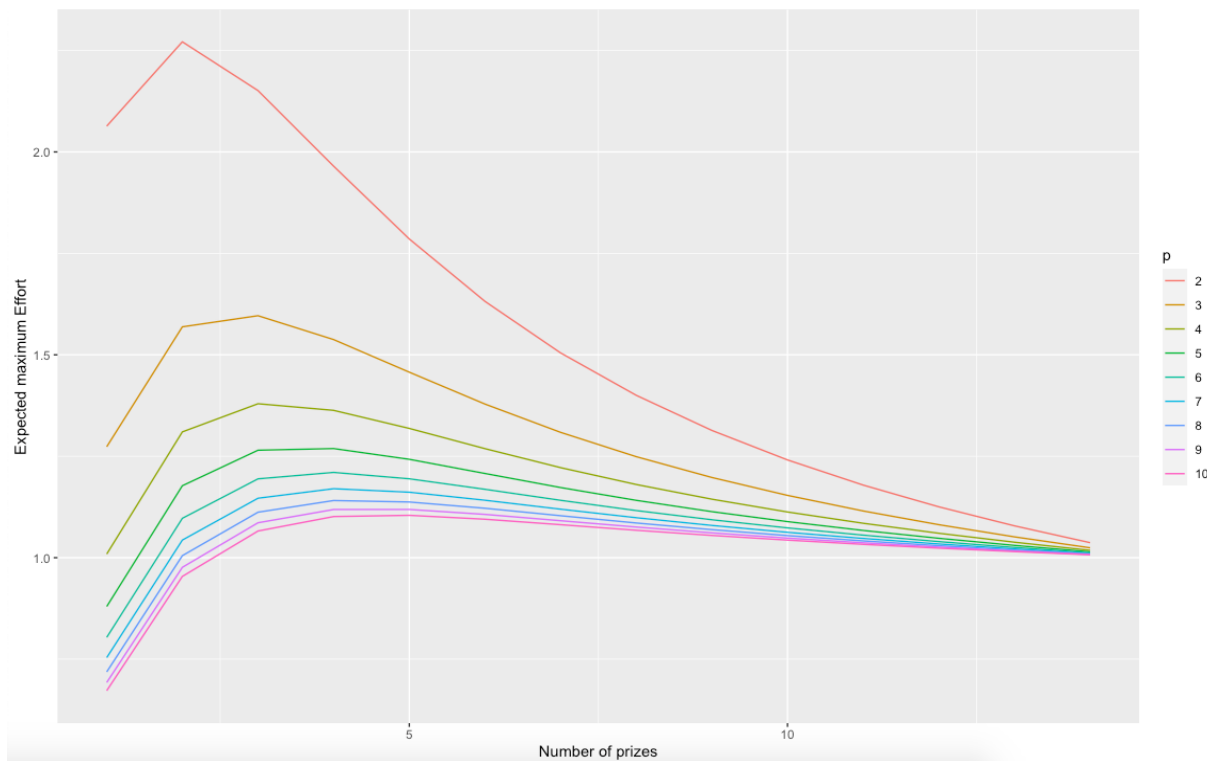


Figure 2: The expected maximum effort as a function of number of prizes for different values of p

4 Screening

In this section, we consider a designer who wants to choose a prize structure to maximize the information revealed by the contest per unit expected effort put in by the agents. For this problem, we'll restrict attention to the simple class of uniform prizes $v = (1, 1, \dots, 1, 0, \dots, 0)$. That is, the designer has to choose a cutoff k so that the k agents who put in the maximum effort are awarded a prize and the remaining $n - k$ agents get 0.

Formally, we have n agents so that the state $\theta \in [0, 1]^n$. Each θ_i is iid according to cdf F and density f . Let $\theta_{(k)}^n$ be the k th order statistic in n samples so that $\theta_{(1)}^n < \theta_{(2)}^n < \dots < \theta_{(n)}^n$. Let $S = \{0, 1\}^n$. Then, a contest that awards prizes to the top k agents generates the signal $s : \Theta \rightarrow S$ such that

$$s_i(\theta) = \begin{cases} 1 & \text{if } \theta_i \leq \theta_{(k)}^n \\ 0 & \text{otherwise} \end{cases}$$

Upon observing an agent with $s_i = 1$, the updated posterior density about the agent's type will be:

$$f_1(t) = \frac{\Pr[s_i = 1 | \theta_i = t] f(t)}{\Pr[s_i = 1]} = \frac{n}{k} \Pr[\theta_{(k)}^{n-1} > t] f(t) = \frac{n}{k} \Pr[X_{(k)}^{n-1} > F(t)] f(t)$$

and the posterior density upon observing $s_i = 0$ will be

$$f_0(t) = \frac{\Pr[s_i = 0 | \theta_i = t] f(t)}{\Pr[s_i = 0]} = \frac{n}{n-k} \Pr[\theta_{(k)}^{n-1} \leq t] f(t) = \frac{n}{n-k} \Pr[X_{(k)}^{n-1} \leq F(t)] f(t)$$

where X_i are $U[0, 1]$ random variables. Let $\mu_i = \int_0^1 t f_i(t) dt$ denote the expected posteriors and again, let $g(\theta)$ denote the equilibrium bidding function. Then, the contest designer's objective is to find the number of prizes to maximize the variance of posterior means (a measure of information revealed) per unit expected effort induced. Formally, it is

$$\max_{k \in \{1, 2, \dots, n-1\}} \frac{\frac{k}{n} (\mu_1 - \mu)^2 + \frac{n-k}{n} (\mu_0 - \mu)^2}{\int_0^1 g(\theta) f(\theta) d\theta}$$

Now we focus on the case where $F(\theta) = \theta^p$ with $p \geq 1$.

Theorem 5. *Suppose $F(\theta) = \theta$ for $\theta \in [0, 1]$. Then, the optimal screening contest with n agents awards $k = \lfloor \frac{n}{2} \rfloor$ prizes.*

The proof is in the appendix. We write it under the prior $F(\theta) = \theta^p$ with $p \geq 1$ and plug in $p = 1$ towards the end to get the result for the uniform case. The proof proceeds by first computing the posteriors using the fact that order statistics of uniform follow the beta distribution. We then find the posterior means and use the expected effort from Theorem 2 to get the value of the objective function. The problem then becomes:

$$\max_{k \in \{1, 2, \dots, n-1\}} \frac{\frac{\mu^2}{k(n-k)} \left(k - n \frac{\beta(k+1 + \frac{1}{p}, n-k)}{\beta(k, n-k)} \right)^2}{\binom{n-1}{k-1} (n-k) \beta\left(k+1 - \frac{1}{p}, n-k\right)}$$

Plugging in $p = 1$ for the uniform case, we get that the optimal is having $\frac{n}{2}$ uniform prizes. The full proof is in the appendix.

Figure 3 illustrates the variance of posterior means per unit expected effort as a function of k under $n = 15$ agents for different p values. We can see that the optimal number of prizes for $p = 1$ is at $k = 8 = \frac{n}{2}$ and for p large, the objective is monotone decreasing in k . In addition, the figure suggests that the optimal number of prizes is weakly decreasing from $n/2$

for $p = 1$ to 1 for p large. We conjecture that this is true more generally. The idea behind the conjecture is that the expected effort is monotone increasing in the number of prizes (2) and this effect dominates the objective for large p (when there are more agents with high marginal costs). Therefore, for large p , awarding just a single prize should be optimal. For smaller $p > 1$, the expected effort is still minimized when there is a single uniform prize but the effect is not big enough to overcome the lack of informativeness with awarding just a single prize. So for $p > 1$ but not too big, a uniform prize structure with prizes between $[1, \frac{n}{2}]$ should be optimal. Further, it seems that the optimal number of prizes is monotone decreasing in p .

Conjecture 2. *In a contest with n agents and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$, the number of prizes in the optimal screening contest (in the class of uniform prize structures) is monotone decreasing in p .*

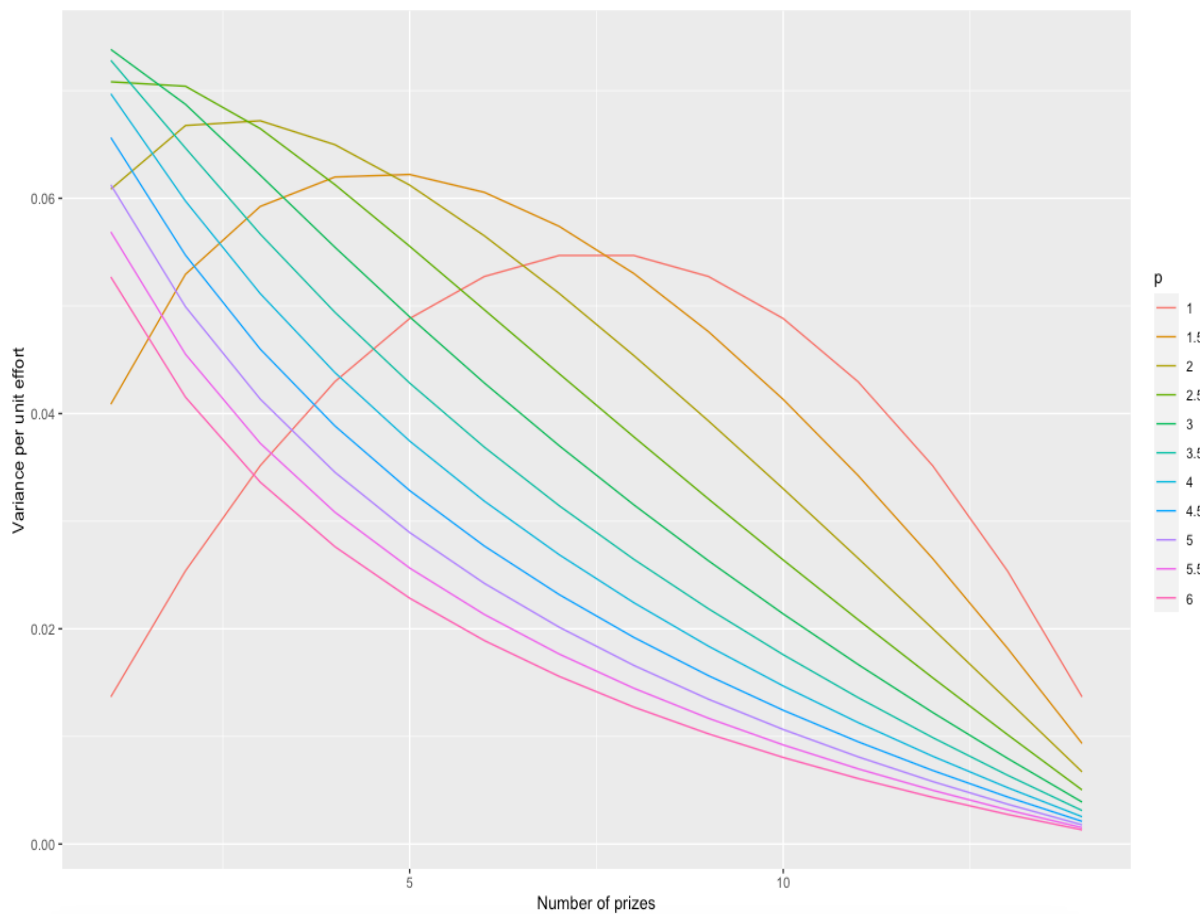


Figure 3: The information revealed per unit effort as a function of number of prizes for different values of p

5 Conclusion

We consider the problem of optimal design of contests as screening devices. In an incomplete information environment with linear effort costs, we find the symmetric Bayes-Nash equilibrium strategy for arbitrary prize structures and show that it is a density function under mild conditions. For the standard objectives of maximizing expected effort, we show that under general conditions, uniform prize structures in which the designer only chooses how many prizes to award are optimal.

For the screening problem, we consider the objective of maximizing the information revealed (measured by variance of posterior means) per unit expected effort put in by the agents. Restricting attention to uniform prize structures for this objective, we find that under the uniform prior, awarding half as many prizes as there are agents is optimal. For the general power distribution $F(\theta) = \theta^p$ with $p > 1$, we conjecture that the number of prizes in the optimal screening contest is decreasing in p and is eventually 1.

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A Proofs for Section 3 (Equilibrium strategy)

Theorem 1. *In a contest with n agents, prizes $v = (1, v_2, \dots, v_{n-1}, 0)$ and prior cdf F , the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g(\theta) = \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt$$

Proof. Suppose $n-1$ agents are playing a monotone decreasing strategy $g(\theta)$. Let $\theta_{(j)}^n$ denote the j th order statistic from n random draws with $\theta_{(0)}^n = 0$ and $\theta_{(n+1)}^n = 1$. Then, an agent of type θ 's utility from putting in x units of effort is given by:

$$\begin{aligned} u(\theta, x) &= \sum_{i=1}^n v_i \Pr[\theta_{(i-1)}^{n-1} \leq g^{-1}(x) \leq \theta_{(i)}^{n-1}] - \theta x \\ &= \sum_{i=1}^n v_i \binom{n-1}{i-1} F(g^{-1}(x))^{i-1} (1 - F(g^{-1}(x)))^{n-i} - \theta x \end{aligned}$$

Now, differentiating with respect to x gives:

$$\begin{aligned} \frac{\partial u(\theta, x)}{\partial x} &= \frac{f(g^{-1}(x))}{g'(g^{-1}(x))} \sum_{i=1}^n v_i \binom{n-1}{i-1} \\ &\quad [(1 - F(g^{-1}(x)))^{n-i} (i-1) F(g^{-1}(x))^{i-2} - F(g^{-1}(x))^{i-1} (n-i) (1 - F(g^{-1}(x)))^{n-i-1}] - \theta \end{aligned}$$

. Setting it to 0 and plugging in $g(\theta) = x$ gives the condition for $g(\theta)$ to be a symmetric Bayes-Nash equilibrium:

$$f(\theta) \sum_{i=1}^n v_i \binom{n-1}{i-1} [(1 - F(\theta))^{n-i} (i-1) F(\theta)^{i-2} - F(\theta)^{i-1} (n-i) (1 - F(\theta))^{n-i-1}] = \theta g'(\theta)$$

An alternate way to write this condition is:

$$-f(\theta) \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1 - F(\theta))^{n-i-1} F(\theta)^{i-1}] = \theta g'(\theta)$$

Using the boundary condition $g(1) = 0$, we get that the symmetric Bayes-Nash equilibrium function is given by

$$\int_{\theta}^1 \frac{f(\theta)}{\theta} \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1 - F(\theta))^{n-i-1} F(\theta)^{i-1}] d\theta$$

Replacing $F(\theta) = t$, we get

$$g(\theta) = \int_{F(\theta)}^1 \frac{1}{F^{-1}(t)} \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1-t)^{n-i-1} t^{i-1}] dt$$

Bringing the summation outside:

$$g(\theta) = \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt$$

Now we check that the second order condition is satisfied. □

Lemma 1. *In a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, 0)$ and prior cdf F , if the equilibrium function is such that $\lim_{\theta \rightarrow 0} \theta g(\theta) = 0$, then it is a density function. That is,*

$$\int_0^1 g(\theta) d\theta = 1$$

Proof. Observe that

$$\int_0^1 \theta g'(\theta) d\theta = \theta g(\theta) \Big|_0^1 - \int_0^1 g(\theta) d\theta = - \int_0^1 g(\theta) d\theta \quad (\text{Assuming } \lim_{\theta \rightarrow 0} \theta g(\theta) = 0)$$

So let's compute this integral $\int_0^1 \theta g'(\theta) d\theta$ given the equilibrium function in 1.

$$\begin{aligned} \int_0^1 \theta g'(\theta) d\theta &= \int_0^1 -f(\theta) \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1-F(\theta))^{n-i-1} F(\theta)^{i-1}] d\theta \\ &= - \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 f(\theta) [(1-F(\theta))^{n-i-1} F(\theta)^{i-1}] d\theta \\ &= - \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 [(1-t)^{n-i-1} t^{i-1}] dt \\ &= - \sum_{i=1}^{n-1} (v_i - v_{i+1}) \\ &= -v_1 + v_n \end{aligned}$$

As a result, we get that for any prize distribution $v_1 \geq v_2 \geq v_3 \cdots \geq v_n$

$$\int_0^1 g(\theta) d\theta = v_1 - v_n$$

With $v_1 = 1, v_n = 0$, we get that the equilibrium function is always a density function. □

Theorem 2. In a contest with n agents and prior cdf $F(\theta) = \theta^p$ with $p > 1$, the expected effort is maximized by $n - 1$ uniform prizes ($v^* = (1, 1, \dots, 1, 0)$).

Proof. Let's look at the expected effort for a general distribution F . We have that

$$\begin{aligned} \int_0^1 g(\theta)f(\theta)d\theta &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \left[\int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1}t^{i-1}]}{F^{-1}(t)} dt \right] f(\theta)d\theta \\ &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1-F(\theta))^{n-i-1}F(\theta)^i}{\theta} f(\theta)d\theta \\ &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1-t)^{n-i-1}t^i}{F^{-1}(t)} dt \end{aligned}$$

The calculation assumes that F is such that $\lim_{\theta \rightarrow 0} \left[\int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1}t^{i-1}]}{F^{-1}(t)} dt \right] F(\theta) = 0$ for all i . This holds, in particular, for the case where $F(\theta) = \theta^p$ with $p \geq 1$ (Lemma 2).

Now let's look at the partial derivative of the expected effort with respect to prize v_i for $i \in \{2, n-1\}$.

Observe that the marginal effect of prize v_i for $i \in \{2, \dots, n-1\}$ is constant and depends only on F . Let's see what exactly it is:

$$\frac{\partial EE(n, v, F)}{\partial v_i} = \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1-t)^{n-i-1}t^i}{F^{-1}(t)} dt - \frac{(n-1)!}{(i-2)!(n-i)!} \int_0^1 \frac{(1-t)^{n-i}t^{i-1}}{F^{-1}(t)} dt$$

Thus, we get that

$$\frac{\partial EE(n, v, F)}{\partial v_i} \geq 0 \iff \frac{n-i}{i-1} \geq \frac{\int_0^1 \frac{(1-t)^{n-i}t^{i-1}}{F^{-1}(t)} dt}{\int_0^1 \frac{(1-t)^{n-i-1}t^i}{F^{-1}(t)} dt}$$

For F uniform, we have that the right hand side equals $\frac{n-i}{i-1}$ and so the marginal gain from increasing or decreasing prizes is 0. Thus, the expected effort is independent of the prize structure.

For $F(\theta) = \theta^p$, we have that the ratio on the right is $\frac{n-i}{i-\frac{1}{p}}$.

So for $p > 1$, the marginal benefit of any prize i is positive. Thus, the optimal prize structure of the form $1 = v_1 \geq v_2 \geq v_3 \cdots \geq v_n = 0$ involves setting $v_i = 1$ for all $i \in \{2, \dots, n-1\}$. In other words, awarding $n-1$ uniform prizes is optimal.

Note that for $p < 1$, the marginal gain is negative but we do not know if the intermediate calculations go through in that case. \square

Theorem 3. In a contest with n agents and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$, the expected minimum effort is maximized by $n - 1$ uniform prizes $v^* = (1, 1, \dots, 1, 0)$.

Proof. Given n , prizes v , and a general F on $[0, 1]$, we know that the minimum effort will be put in by the agent with the highest marginal cost.

Let $\theta_m = \max\{\theta_1, \dots, \theta_n\}$ and let F_m denote the cdf of θ_m . Then, we know that $F_m(t) = \mathbb{P}[\theta_m \leq t] = F(t)^n$ and so the density is $f_m(t) = nF(t)^{n-1}f(t)$

Thus, from Theorem 1, the expected minimum effort is given by:

$$\begin{aligned} \int_0^1 g(\theta) f_m(\theta) d\theta &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \left[\int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt \right] f_m(\theta) d\theta \\ &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \left[\int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt \right] nF(\theta)^{n-1} f(\theta) d\theta \\ &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1-F(\theta))^{n-i-1} F(\theta)^{n+i-1}}{\theta} f(\theta) d\theta \\ &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1-t)^{n-i-1} t^{n+i-1}}{F^{-1}(t)} dt \end{aligned}$$

The calculation assumes that F is such that $\lim_{\theta \rightarrow 0} \left[\int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt \right] F(\theta)^n = 0$ for all i . This holds, in particular, for the case where $F(\theta) = \theta^p$ with $p \geq 1$ (Lemma 2).

Again, the expected minimum effort is linear in prizes and we can show that

$$\frac{\partial EMINE(n, v, F)}{\partial v_i} = \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1-t)^{n-i-1} t^{n+i-1}}{F^{-1}(t)} dt - \frac{(n-1)!}{(i-2)!(n-i)!} \int_0^1 \frac{(1-t)^{n-i} t^{n+i-2}}{F^{-1}(t)} dt$$

Thus, we get that

$$\frac{\partial EMINE(n, v, F)}{\partial v_i} \geq 0 \iff \frac{n-i}{i-1} \geq \frac{\int_0^1 \frac{(1-t)^{n-i} t^{n+i-2}}{F^{-1}(t)} dt}{\int_0^1 \frac{(1-t)^{n-i-1} t^{n+i-1}}{F^{-1}(t)} dt}$$

For F uniform, we have that the right hand side equals $\frac{n-i}{n+i-2}$ and so the marginal gain from increasing prize i is > 0 . Thus, the expected effort is maximized at $n - 1$ uniform prizes.

For $F(\theta) = \theta^p$, we have that the ratio on the right is $\frac{n-i}{n+i-1-\frac{1}{p}}$.

So for $p > 1$, the marginal benefit of any prize i is positive. Thus, the optimal prize structure of the form $1 = v_1 \geq v_2 \geq v_3 \dots \geq v_n = 0$ involves setting $v_i = 1$ for all $i \in \{2, \dots, n-1\}$. In other words, awarding $n - 1$ uniform prizes is optimal. \square

Theorem 4. In a contest with n agents and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$, the expected maximum effort is maximized at some uniform prize structure $v = (1, \dots, 1, 0, \dots, 0)$.

Proof. Given n , prizes v , and a general F on $[0, 1]$, we know that the maximum effort will be put in by the agent with the lowest marginal cost.

Let $\theta_i = \min\{\theta_1, \dots, \theta_n\}$ and let F_i denote the cdf of θ_i . Then, we know that $F_i(t) = \mathbb{P}[\theta_i \leq t] = 1 - (1 - F(t))^n$ and so the density is $f_i(t) = n(1 - F(t))^{n-1}f(t)$

Thus, from Theorem 1, the expected maximum effort is given by:

$$\begin{aligned} \int_0^1 g(\theta) f_i(\theta) d\theta &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \left[\int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1}t^{i-1}]}{F^{-1}(t)} dt \right] f_i(\theta) d\theta \\ &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \left[\int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1}t^{i-1}]}{F^{-1}(t)} dt \right] n(1 - F(\theta))^{n-1} f(\theta) d\theta \\ &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1 - F(\theta))^{n-i-1} F(\theta)^{i-1}}{\theta} f(\theta) (1 - (1 - F(\theta))^n) d\theta \\ &= \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1-t)^{n-i-1}t^{i-1}}{F^{-1}(t)} (1 - (1-t)^n) dt \end{aligned}$$

The calculation assumes that F is such that $\lim_{\theta \rightarrow 0} \left[\int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1}t^{i-1}]}{F^{-1}(t)} dt \right] (1 - (1 - F(\theta))^n) = 0$ for all i . This holds, in particular, for the case where $F(\theta) = \theta^p$ with $p \geq 1$ (Lemma 2).

Again, the expected maximum effort is linear in prizes and we can show that

$$\begin{aligned} \frac{\partial EMAXE(n, v, F)}{\partial v_i} &= \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_0^1 \frac{(1-t)^{n-i-1}t^{i-1}}{F^{-1}(t)} (1 - (1-t)^n) dt - \\ &\quad \frac{(n-1)!}{(i-2)!(n-i)!} \int_0^1 \frac{(1-t)^{n-i}t^{i-2}}{F^{-1}(t)} (1 - (1-t)^n) dt \end{aligned}$$

Thus, we get that

$$\frac{\partial EMAXE(n, v, F)}{\partial v_i} \geq 0 \iff \frac{n-i}{i-1} \geq \frac{\int_0^1 \frac{(1-t)^{n-i}t^{i-2}}{F^{-1}(t)} (1 - (1-t)^n) dt}{\int_0^1 \frac{(1-t)^{n-i-1}t^{i-1}}{F^{-1}(t)} (1 - (1-t)^n) dt}$$

Let's simplify the inequality on the right for the case where $F(\theta) = \theta^p$ with $p \geq 1$.

$$\frac{n-i}{i-1} \geq \frac{\int_0^1 (1-t)^{n-i}t^{i-2-\frac{1}{p}} (1 - (1-t)^n) dt}{\int_0^1 (1-t)^{n-i-1}t^{i-1-\frac{1}{p}} (1 - (1-t)^n) dt}$$

$$\begin{aligned}
&= \frac{\beta(i-1-\frac{1}{p}, n-i+1) - \beta(i-1-\frac{1}{p}, 2n-i+1)}{\beta(i-\frac{1}{p}, n-i) - \beta(i-\frac{1}{p}, 2n-i)} \\
&= \frac{\Gamma(i-1-\frac{1}{p})}{\Gamma(i-\frac{1}{p})} \frac{\frac{(n-i)!}{\Gamma(n-\frac{1}{p})} - \frac{(2n-i)!}{\Gamma(2n-\frac{1}{p})}}{\frac{(n-i-1)!}{\Gamma(n-\frac{1}{p})} - \frac{(2n-i-1)!}{\Gamma(2n-\frac{1}{p})}} \\
&= \frac{1}{(i-1-\frac{1}{p})} \frac{\frac{(n-i)!}{\Gamma(n-\frac{1}{p})} \left(1 - \prod_{j=1}^n \frac{n-i+j}{n-\frac{1}{p}-1+j}\right)}{\frac{(n-i-1)!}{\Gamma(n-\frac{1}{p})} \left(1 - \prod_{j=1}^n \frac{n-i-1+j}{n-1-\frac{1}{p}+j}\right)} \\
&= \frac{n-i}{i-1-\frac{1}{p}} \frac{\left(1 - \prod_{j=1}^n \frac{n-i+j}{n-\frac{1}{p}-1+j}\right)}{\left(1 - \prod_{j=1}^n \frac{n-i-1+j}{n-1-\frac{1}{p}+j}\right)}
\end{aligned}$$

Thus, the condition simplifies to:

$$\frac{\partial EMAXE(n, v, F)}{\partial v_i} \geq 0 \iff \frac{i-1-\frac{1}{p}}{i-1} \geq \frac{\left(1 - \prod_{j=1}^n \frac{n-i+j}{n-1-\frac{1}{p}+j}\right)}{\left(1 - \prod_{j=1}^n \frac{n-i+j-1}{n-1-\frac{1}{p}+j}\right)}$$

□

B Proofs for Section 4 (Screening)

Theorem 5. *Suppose $F(\theta) = \theta$ for $\theta \in [0, 1]$. Then, the optimal screening contest with n agents awards $k = \lfloor \frac{n}{2} \rfloor$ prizes.*

Proof. We will do the calculations assuming $F(\theta) = \theta^p$. We'll plug in $p = 1$ towards the end to get the result for the uniform case. For the case where $F(\theta) = \theta^p$ (with $p \geq 1$), we have that the posteriors are

$$f_1(t) = \frac{n}{k} \Pr[X_{(k)}^{n-1} > t^p] p t^{p-1}$$

and

$$f_0(t) = \frac{n}{n-k} \Pr[X_{(k)}^{n-1} \leq t^p] p t^{p-1}$$

We know that if $X_{(k)}^n$ is the k th order statistic in n uniform random samples then its distribution is $\beta(k, n+1-k)$. Thus,

$$\Pr(X_{(k)}^n \leq t) = I_t(k, n+1-k) = \frac{\beta_t(k, n+1-k)}{\beta(k, n+1-k)} = \frac{n!}{(k-1)!(n-k)!} \int_0^t x^{k-1} (1-x)^{n-k} dx$$

A useful object in studying this case will be

$$\begin{aligned}
\int_0^1 t^{\frac{1}{p}} I_t(k, n+1-k) dt &= \frac{1}{\beta(k, n+1-k)} \int_0^1 t^{\frac{1}{p}} \left[\int_0^t x^{k-1} (1-x)^{n-k} dx \right] dt \\
&= \frac{1}{\beta(k, n+1-k)} \left[\frac{p\beta(k, n-k+1)}{1+p} - \frac{p\beta(k+1+\frac{1}{p}, n-k+1)}{1+p} \right] \\
&= \frac{p}{1+p} \left(1 - \frac{\beta(k+1+\frac{1}{p}, n-k+1)}{\beta(k, n+1-k)} \right) \\
&= \frac{p}{1+p} \left(1 - \frac{\Gamma(k+1+\frac{1}{p})\Gamma(n+1)}{\Gamma(n+2+\frac{1}{p})\Gamma(k)} \right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mu_1 &= \int_0^1 t f_1(t) dt \\
&= \frac{n}{k} \int_0^1 \Pr[X_{(k)}^{n-1} > t^p] p t^p dt \\
&= \frac{n}{k} \int_0^1 \left(1 - \Pr[X_{(k)}^{n-1} \leq t^p] \right) p t^p dt \\
&= \frac{n}{k} \frac{p}{p+1} - \frac{np}{k} \int_0^1 \Pr[X_{(k)}^{n-1} \leq t^p] t^p dt \\
&= \frac{n}{k} \frac{p}{p+1} - \frac{n}{k} \int_0^1 \Pr[X_{(k)}^{n-1} \leq z] z^{\frac{1}{p}} dz \\
&= \frac{n}{k} \mu z \quad \left(z = \frac{\beta(k+1+\frac{1}{p}, n-k)}{\beta(k, n-k)} = \frac{(kp+1)\Gamma(k+\frac{1}{p})\Gamma(n)}{(np+1)\Gamma(n+\frac{1}{p})\Gamma(k)} \right)
\end{aligned}$$

Similarly, we get

$$\mu_0 = \int_0^1 t f_0(t) dt = \frac{np}{(n-k)(1+p)} \left(1 - \left(\frac{k}{n} \right)^{\frac{p+1}{p}} \right) = \frac{n}{n-k} \mu (1-z)$$

Plugging the values into our measure for information revealed, we get

$$\begin{aligned}
\frac{k}{n} (\mu_1 - \mu)^2 + \frac{n-k}{n} (\mu_0 - \mu)^2 &= \frac{1}{kn} \mu^2 (nz - k)^2 + \frac{1}{(n-k)n} \mu^2 (k - nz)^2 \\
&= \mu^2 \frac{(nz - k)^2}{n} \left(\frac{1}{k} + \frac{1}{n-k} \right)
\end{aligned}$$

$$= \mu^2 \frac{(nz - k)^2}{k(n - k)}$$

Using the expression for expected effort from Theorem 2, we get that the designer's objective in this case takes the form:

$$\max_{k \in \{1, 2, \dots, n-1\}} \frac{\frac{\mu^2}{k(n-k)} \left(k - n \frac{\beta(k+1 + \frac{1}{p}, n-k)}{\beta(k, n-k)} \right)^2}{\binom{n-1}{k-1} (n-k) \beta\left(k+1 - \frac{1}{p}, n-k\right)}$$

For the uniform case with $p = 1$, this simplifies to

$$\mu^2 \frac{k(n-k)}{(n+1)^2}$$

This is clearly maximized at $k = \frac{n}{2}$ and so, we get that awarding $\frac{n}{2}$ prizes is optimal when the contest has n participants with uniform marginal costs. □