An efficiency ordering of k-price auctions under complete information

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Abstract

We study k-price auctions in a complete information environment and characterize all pure-strategy Nash equilibrium outcomes. In a setting with n agents having ordered valuations, we show that any agent, except those with the lowest $k-2$ valuations, can win in equilibrium. As a consequence, worst-case welfare increases monotonically as we go from second-price auction ($k = 2$) to lowest-price auction ($k = n$), with the first-price auction achieving the highest worst-case welfare.

1 Introduction

We study k-price sealed-bid auctions in a complete information environment with n agents who have strictly ordered valuations. In a k-price auction, all n agents submit their bids, the highest bidder wins the object (with ties broken in favor of the agent with the highest valuation), and pays the kth highest bid. We fully characterize the set of pure-strategy Nash equilibrium outcomes for every k-price auction.

For $k \in \{2, \dots, n\}$, we show that any of the top $n-(k-2)$ valuation agents can win in equilibrium, while the bottom $(k-2)$ agents can never win. In other words, the second-price auction can be won by any of the n agents, the third-price auction by any of the top $n-1$ agents, and so on, until we reach the lowest-price auction ($k = n$), which can only be won by the top two agents. We further show that in the first-price auction ($k = n+1$), only the top agent can win. This equilibrium characterization reveals a natural ordering of k-price auctions in terms of their worst-case allocative efficiency: the worst-case equilibrium allocation becomes strictly more efficient as k increases from 2 to $n+1$.

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In closely related work, Tauman (2002) and Mathews and Schwartz (2017) also study k-price auctions in complete information environments. Under the restriction to pure strategies that are not weakly dominated, Tauman (2002) shows that for any k-price auction, only the top agent can win in equilibrium. Subsequently, Mathews and Schwartz (2017) construct an equilibrium in mixed-strategies where the top agent does not win. In comparison, we characterize all pure-strategy Nash equilibrium outcomes for all k-price auctions and obtain an ordering of these auctions based on their worst-case allocative efficiency.[1]

2 Model

A seller is selling an indivisible object to a set $N = \{1, \ldots, n\}$ of agents. Each agent $i \in N$ has a valuation $v_i > 0$ for the object, and we assume that

$$v_1 > v_2 > \cdots > v_n,$$

with $v_{n+1} = 0$. The valuations are common knowledge.

The object is sold using a sealed-bid k-price auction. Each agent $i \in N$ simultaneously submits a non-negative bid $b_i \in \mathbb{R}_+$. The object is awarded to the agent who submits the highest bid, with ties resolved in favor of the agent with the highest valuation. The winner pays the kth highest bid, denoted $b_{(k)}$, and all other agents pay zero. The utility of agent $i \in N$ at bid profile $b = (b_1, \ldots, b_n)$ is

$$u_i(b) = \begin{cases} v_i - b_{(k)} & \text{if } i = \min\{j \in N : b_j = b_{(1)}\}, \\ 0 & \text{otherwise.} \end{cases}$$

We characterize the pure-strategy Nash equilibrium outcomes of the k-price auction for all $k \in \{1, 2, \ldots, n\}$, where $k = 1$ (also $n + 1$) denotes the first-price auction, $k = 2$ the second-price auction, and $k = n$ the lowest-price auction. Since we focus on pure strategies, our characterization depends only on the ordinal ranking over bid profiles induced by u_i.[2] Moreover, the characterization holds under any tie-breaking rule for $k \in \{2, \ldots, n\}$, and the choice matters only in the case of the first-price auction.[3]

[1] Other work on k-price auctions has focused on incomplete information settings (Kagel and Levin (1993), Monderer and Tennenholtz (2000, 2004), Mezzetti and Tsetlin (2009), Azrieli and Levin (2012), Mihelich and Shu (2020), Skitmore (2014)).

[2] Formally, we can define an outcome set $X = \mathbb{R}_+ \cup \{-1\}$ for each agent, where $x \in \mathbb{R}_+$ represents the payment made when it wins the object, and $x = -1$ denotes not winning. An agent $i \in N$ with valuation $v_i > 0$ is represented by a preference relation $\succ_i$ over X, such that $x \succ_i y$ for all $x < y \in \mathbb{R}_+$, and $-1 \sim_i v_i$. The equilibrium characterization in Proposition 1 holds as long as u_i is consistent with $\succ_i$ for all $i \in N$.

[3] Previous work has typically assumed that ties are broken uniformly at random (Tauman (2002), Mathews and Schwartz (2017)). Our assumption allows us to discuss the first-price auction alongside the other k-price auctions in a unified way, aids exposition more generally, and is natural given our focus on welfare. Additionally, from the seller's perspective, implementing this tie-breaking rule requires only knowledge of the ordinal ranking of agents' valuations, not their exact values.

3 Results

Our main result shows that under the k-price auction, any agent whose valuation ranks among the top $n-(k-2)$ can win the object at any price between the $n-(k-3)$ highest valuation and their own valuation, and no other outcomes can arise in equilibrium.

Proposition 1. *Consider a k-price auction with $k \in \{2, \dots, n+1\}$. For any $i \in \{1, \dots, n\}$ and any $p \in \mathbb{R}_+$, there exists a pure-strategy Nash equilibrium in which agent i wins and pays p if and only if*

$$v_i > v_{n-(k-3)} \text{ and } p \in [v_{n-(k-3)}, v_i].$$

Proof. We first prove the forward direction. Suppose b is an equilibrium bid profile where agent i wins and pays p. We begin by showing that the seller's revenue $p \geq v_{n-(k-3)}$.

Lemma 1. $p \geq v_{n-(k-3)}$.

Proof. Suppose towards a contradiction that $p = b_{(k)} < v_{n-(k-3)}$.

1. $k = 2$: For the second-price auction, $p < v_{n-(k-3)}$ simplifies to $p < v_{n+1} = 0$. But $p = b_{(2)} \geq 0$, which is a contradiction.

2. $k \in \{3, \dots, n\}$: For this k-price auction, $p < v_{n-(k-3)}$ and $p = b_{(k)}$ imply that

   (a) At least $(n-(k-3))$ agents have valuation $> p$ (as $v_1 > \dots > v_{n-(k-3)} > p$).

   (b) At least k agents bid $\geq p$ at profile b.

   It follows then that there are at least three distinct agents with valuations strictly greater than p, each bidding at least p. At least two of these three agents receive utility 0. At least one of these two agents can raise their bid above $b_{(1)}$ without changing the price, thereby receiving strictly positive utility. This contradicts b being an equilibrium.

3. $k = n+1$: For the first-price auction, $p < v_{n-(k-3)}$ simplifies to $p < v_2$. Notice that at least one of agents 1 and 2 receive utility 0, and this agent can deviate by bidding in $(b_{(1)}, v_2)$, and receive strictly positive utility. This contradicts b being an equilibrium.

It follows that $p \geq v_{n-(k-3)}$. □

We now show that the winner's valuation $v_i > v_{n-(k-3)}$.

Lemma 2. $v_i > v_{n-(k-3)}$.

Proof. Suppose towards a contradiction that $v_i \leq v_{n-(k-3)}$. From Lemma 1, agent i must pay a price $p \geq v_{n-(k-3)}$. Since i's utility must be non-negative (as it can deviate to bidding 0 otherwise), the only possibility is that $v_i = p = v_{n-(k-3)}$.

1. $k = 2$: For the second-price auction, $v_i = p = v_{n-(k-3)}$ simplifies to $v_i = p = v_{n+1} = 0$. But $v_i > 0$ for all $i \in N$, which is a contradiction.

2. $k \in \{3, \ldots, n\}$: For this k-price auction, $v_i = p = v_{n-(k-3)}$ and $p = b_{(k)}$ imply that

   (a) At least $(n - (k-2))$ agents have valuation $> p$ (as $v_1 > \cdots > v_{n-(k-2)} > p$).

   (b) At least k agents bid $\geq p$ at profile b.

   It follows then that there are at least two distinct agents with valuations strictly greater than p, each bidding at least p. Further, both these agents receive utility 0, and at least one of these two agents can deviate by bidding $> b_{(1)}$, and receive strictly positive utility. This contradicts b being an equilibrium.

3. $k = n + 1$: For the first-price auction, $v_i = p = v_{n-(k-3)}$ simplifies to $v_i = p = v_2$. But then, agent 1 receives a utility of 0, and it can deviate by bidding in the interval (v_2, v_1), and receive strictly positive utility. This contradicts b being an equilibrium.

Thus, $v_i > v_{n-(k-3)}$. □

Together, it must be that winner's valuation $v_i > v_{n-(k-3)}$ and the price $p \in [v_{n-(k-3)}, v_i]$.

We now prove the reverse direction. Suppose $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_i]$. We construct an equilibrium profile where agent i wins and pays p.

1. $k \in \{2, \ldots, n\}$: For this k-price auction, consider the bid profile b where

$$b_i = v_1 \text{ and } b_{-i} = (\underbrace{p, \ldots, p}_{n-(k-1) \text{ agents}}, \underbrace{v_1, \ldots, v_1}_{(k-2) \text{ agents}}).$$

   At profile b, agent i wins the k-price auction (as ties are broken in favor of agent with highest valuation), and pays a price $b_{(k)} = p$ for the good. Agent i's utility is $v_i - p \geq 0$ and for $j \neq i$, agent j's utility is 0. We now verify that b is indeed a Nash equilibrium. Consider agent $j \in N$.

   (a) $j = i$: If $b'_i \geq v_1$, agent i's utility does not change. If $b'_i < v_1$, agent i's utility is either 0 or does not change (possible when $k = 2$).

   (b) $j \in \{1, \ldots, n - (k-2)\} \setminus \{i\}$: If $b'_j > v_1$, agent j's utility will be $v_j - v_1 \leq 0$. If $b'_j \leq v_1$, agent j's utility will be either 0 or < 0.

   (c) $j \in \{n - (k-3), \ldots, n\}$: If $b'_j > v_1$, agent j's utility will be $v_j - p \leq 0$. If $b'_j \leq v_1$, agent j's utility remains 0.

   Thus, no agent has a profitable deviation, and b is an equilibrium in the k-price auction.

2. $k = n + 1$: For the first-price auction, $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_1]$ simplify to $v_i > v_2$ and $p \in [v_2, v_1]$. Consider the bid profile

$$b = (p, \ldots, p).$$

   It is straightforward to verify that b is a Nash equilibrium in the first-price auction.

Thus, there exists a Nash equilibrium where agent i wins and pays p.[4] □

Proposition 1 yields a ranking of k-price auctions in terms of worst-case welfare. Let

$$\underline{W}^k = \min\{v_i : \exists b \text{ such that } b \text{ is an equilibrium of the } k\text{-price auction where agent } i \text{ wins}\}.$$

Corollary 1. *For $k \in \{2, \ldots, n+1\}$, the worst-case equilibrium welfare of k-price auction is*

$$\underline{W}^k = v_{n-(k-2)}.$$

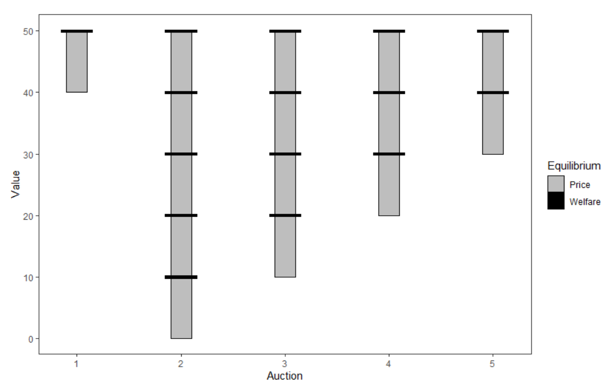
Hence,

$$\underline{W}^2 < \underline{W}^3 < \cdots < \underline{W}^n < \underline{W}^{n+1}.$$

Thus, the second-price auction is the worst, as even the lowest-valuation agent can win, while higher values of k progressively exclude lower-valuation agents from winning in equilibrium, culminating in the first-price auction, where only the highest-valuation agent can win.[5]

Lastly, we illustrate our results through an example with $n = 5$ agents whose valuations are $v_1 = 50, v_2 = 40, v_3 = 30, v_4 = 20$ and $v_5 = 10$. In figure 1 the black bars show the possible equilibrium welfare, while the grey vertical bars show the interval of possible equilibrium prices. In the first-price auction, only agent 1 can win at a price in $[40, 50]$. In the second-price auction, any agent i can win at a price in $[0, v_i]$. And so on, until we reach the lowest-price auction, where either agent 2 can win at a price in $[30, 40]$ or agent 1 can win at a price in $[30, 50]$. For example, under the lowest-price auction, $b = (35, 70, 65, 60, 55)$ is an equilibrium profile in which agent 2 wins (leading to welfare of 40) at price $p = 35$.

Figure 1: k-Price Auctions With 5 Agents



[4]For $k \in \{2, \ldots, n\}$, our construction can be modified by setting $b_i > v_1$ so that the resulting profile remains a Nash equilibrium with the same outcome, regardless of the tie-breaking rule. For the first-price auction, however, such modifications need not exist, and the result relies on our assumed tie-breaking rule.

[5]From Proposition 1, the same ordering also emerges when ranking k-price auctions by worst-case revenue.

References

Azrieli, Y. and D. Levin (2012): "Dominance Solvability of Large k-Price Auctions," *The BE Journal of Theoretical Economics*, 12. 2

Kagel, J. H. and D. Levin (1993): "Independent private value auctions: Bidder behaviour in first-, second-and third-price auctions with varying numbers of bidders," *The Economic Journal*, 103, 868–879. 2

Mathews, T. and J. A. Schwartz (2017): "A note on k-price auctions with complete information when mixed strategies are allowed," *Economics Letters*, 153, 6–8. 2

Mezzetti, C. and I. Tsetlin (2009): "Auctions in which losers set the price," *Games and Economic Behavior*, 66, 855–864. 2

Mihelich, M. and Y. Shu (2020): "Analytical solution of k th price auction," *International Journal of Game Theory*, 49, 875–884. 2

Monderer, D. and M. Tennenholtz (2000): "K-price auctions," *Games and Economic Behavior*, 31, 220–244. 2

——— (2004): "K-price auctions: Revenue inequalities, utility equivalence, and competition in auction design," *Economic Theory*, 24, 255–270. 2

Skitmore, M. (2014): "km th price sealed-bid auctions with general independent values and equilibrium linear mark-ups," *Journal of the operational research society*, 65, 1864–1875. 2

Tauman, Y. (2002): "A note on k-price auctions with complete information," *Games and Economic Behavior*, 41, 161–164. 2